

STRONG STATIONARY DUALITY FOR DIFFUSION PROCESSES

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ABSTRACT. We develop the theory of strong stationary duality for diffusion processes on finite intervals. We analytically derive the generator and boundary behavior of the dual process and recover a central tenet of the classical theory by proving that the separation mixing time in the primal diffusion is equal in law to the absorption time in the dual diffusion. We also exhibit our strong stationary dual as the natural limiting process of the strong stationary dual sequence of a well chosen sequence of approximating birth-and-death Markov chains, allowing for simultaneous numerical simulations of our primal and dual diffusion processes. Lastly, we show how our new definition of diffusion duality allows the spectral theory of cutoff phenomena to extend naturally from birth-and-death Markov chains to the present diffusion context.

1. INTRODUCTION AND BACKGROUND

Strong stationary duality (SSD)—first developed in the setting of discrete-state Markov chains in [4] and [9]—has proven to be a powerful tool in the study of mixing times of Markov chains. In the Markov chain setting, strong stationary duality guarantees that the separation mixing time in a Markov chain is equal in law to the absorption time in a suitably defined dual chain. By studying and bounding the absorption time, which is often more tractable than direct consideration of the mixing time, one can tightly bound the separation mixing time in the primal chain. This duality between hitting times and mixing times plays a leading role in the development of such diverse techniques as perfect sampling of Markov chains (see [10], [13]), characterizations of separation cut-offs in birth and death chains (see [6]), stochastic constructions of Markov chain hitting times (see [11], [5], [14]), and the analysis of the fastest mixing Markov chain on a graph (see [12]), to name a few.

However, since initially being referenced in [9], extending SSD from Markov chains to the diffusion regime has remained an open problem. Herein, we present a major step towards this extension. Utilizing a functional analytic approach, in Section 3.1 we systematically develop the theory of SSD for diffusion processes on finite intervals and analytically derive the form of the dual diffusion’s generator; in the process, we also explicitly derive the boundary behavior of the dual diffusion. We further motivate our definition in Section 4 by showing that a suitably defined sequence of Markov chains and their strong stationary duals converge, respectively, to our primal diffusion and its strong stationary dual. In Section 5, we recover

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a central tenet of the classical Markov chain theory in our diffusion setting by proving that the separation mixing time in the primal diffusion is equal in law to the absorption time in the dual diffusion. In Section 6, we exploit this connection to derive the analogue to the birth-and-death cut-off phenomenon theory of [6] in the diffusion setting.

Recently, and independently of our work, wonderful developments in diffusion strong stationary duality have been made in [24] and [25]. Our present work was originally presented in the dissertation of the second author [23] certified in December 2012, and predates the work of [24] and [25].

2. BACKGROUND

Let X be a time homogeneous one-dimensional diffusion process defined on the (possibly infinite) real interval I with infinitesimal generator

$$(2.1) \quad A = \frac{1}{2}b(x)\frac{d^2}{dx^2} + a(x)\frac{d}{dx}.$$

Denoting the closure of I by \bar{I} and the interior of I by $I^\circ = (l, r)$, to avoid pathologies, we shall assume throughout that $a(\cdot), b(\cdot) \in C(I^\circ)$ and $b > 0$ on I° .

Denote the speed measure of X by M and its density by m , and denote the scale function of X by S . Feller classified the boundary behavior of X at l (analogous results holding at r if $r \in I$) by looking at the behavior of

$$N(l) := \int_{(l,x]} [S(x) - S(\eta)] M(d\eta), \quad \Sigma(l) := \int_{(l,x]} [M(x) - M(\eta)] dS(\eta)$$

for a fixed $x \in I^\circ$ and by calculating boundary conditions satisfied by elements of the domain of A , which we shall call \mathcal{D}_A (see [8, Section 8.1] for more details). Entrance boundaries are characterized by $N(l) < \infty, \Sigma(l) = \infty$. Note that [18, Section 15.6] implies that to show l is entrance, it suffices to show that $N(l) < \infty$ and $S(l, x) = \lim_{y \downarrow l} [S(x) - S(y)] = \infty$. Exit boundaries are characterized by $N(l) = \infty, \Sigma(l) < \infty$. Natural boundaries are characterized by $N(l) = \infty$ and $\Sigma(l) = \infty$. Finally, regular boundaries are characterized by $N(l) < \infty, \Sigma(l) < \infty$. The behavior of the diffusion at a regular boundary will be characterized by boundary conditions satisfied by elements $f \in \mathcal{D}_A$. In particular, we say that l is *instantaneously reflecting* if $f \in \mathcal{D}_A$ implies that

$$\frac{df}{dS}^+(l) = \lim_{x \downarrow l} \frac{df}{dS}(x) = \lim_{x \downarrow l} \frac{f'(x)}{s(x)} = 0.$$

We say l is *absorbing* if $f \in \mathcal{D}_A$ implies that $(Af)(l) = 0$.

Presently and in the sequel, let X be a regular diffusion process on a finite closed interval $I (= [0, 1])$, without loss of generality) with initial distribution π_0 and generator A . Assume that 0 and 1 are instantaneously reflecting boundaries X . The boundary behavior of X guarantees that M is a finite measure on I° , and normalizing $M(dx)$ to a probability measure gives the unique invariant distribution of X , which we will denote by $\Pi(dx)$. As with M , for arbitrary $c \in I^\circ$, let us adopt the shorthand $\Pi(x) := \int_{y=c}^x \pi(y) dy$, where π is the density for Π with respect to Lebesgue measure, and note that regularity of X guarantees $\pi > 0$ on I° . The reflecting behavior at 0 guarantees $\lim_{c \downarrow 0} \int_{y=c}^x \pi(y) dy$ exists and is finite for all $x \in I^\circ$, and so to ease notation we may let $\Pi(x) = \int_{y=0}^x \pi(y) dy$ defined as an improper integral. Lastly, let $(P_t)_{t=0}^\infty$ be the Markov transition function associated

with X and denote the corresponding transition densities with respect to Lebesgue measure by $(p_t)_{t=0}^\infty$.

Based on the boundary behavior of X , we can completely specify the domain of A as

$$(2.2) \quad \mathcal{D}_A = \left\{ f \in C(I) \cap C^2(I^\circ) \mid Af \in C(I), \frac{df}{dS}^+(0) = \frac{df}{dS}^-(1) = 0 \right\}$$

(see [8, Section 8.1], especially (1.11) there, with $q_0 = 0 = q_1$ because both boundaries are instantaneously reflecting), where as above

$$\frac{df}{dS}^+(0) = \lim_{x \downarrow 0} \frac{df}{dS}(x) = \lim_{x \downarrow 0} \frac{f'(x)}{s(x)},$$

and

$$\frac{df}{dS}^-(1) = \lim_{x \uparrow 1} \frac{df}{dS}(x) = \lim_{x \uparrow 1} \frac{f'(x)}{s(x)}.$$

Let $F[0, 1]$ be the space of bounded real valued measurable functions on $[0, 1]$ equipped with its usual Borel σ -field \mathcal{B} . Let $M[0, 1]$ be the space of signed measures on $([0, 1], \mathcal{B})$. As in [21, Section 7.1], we note the natural bilinear functional on $F[0, 1] \times M[0, 1]$ defined by $(\mu, f) = \int_0^1 f(x) \mu(dx)$. We denote the adjoint of the operator T_t (with respect to this functional) by U_t , where $(T_t)_{t=0}^\infty$ is the one parameter Markov semigroup associated with $(P_t)_{t=0}^\infty$.

Note that $a(\cdot)$, $b(\cdot)$, $M(\cdot)$, and $\pi(\cdot)$ are defined only on I° . For notational convenience, any expressions involving these functions and ∂I are to be interpreted as the corresponding limiting expression (when such a limit exists!). For example, for $0 < x < 1$ we shall write the improper integral $\int_0^x f(y) \pi(y) dy$ rather than the equivalent $\lim_{z \downarrow 0} \int_z^x f(y) \pi(y) dy$.

3. STRONG STATIONARY DUALITY FOR DIFFUSIONS

3.1. Definition of the strong stationary dual. Let X^* be a second (Feller) diffusion process on I with initial distribution π_0^* and generator A^* . As in the continuous-time discrete-state Markov chain setting (see [9]), we define the notion of algebraic duality between X and X^* :

Definition 3.1. Consider the integral operator Λ acting on $F[0, 1]$ defined by

$$(\Lambda f)(x) := \begin{cases} \int_0^x \pi^{(x)}(y) f(y) dy & \text{if } x > 0, \\ f(0) & \text{if } x = 0, \end{cases}$$

where we define the kernel

$$\pi^{(x)}(y) := \frac{\pi(y)}{\Pi(x)} \quad 0 < y \leq x < 1, \quad \text{and } \pi^{(1)} \equiv \pi.$$

We say that X^* is a strong stationary dual of X if

$$(3.1) \quad \Lambda \text{ maps } \mathcal{D}_A \text{ into } \mathcal{D}_{A^*}$$

and

$$(3.2) \quad \Lambda A = A^* \Lambda \text{ as operators defined on } \mathcal{D}_A$$

and

$$(3.3) \quad (\pi_0, f) = (\pi_0^*, \Lambda f) \text{ for all } f \in F[0, 1].$$

Remark 3.2. If $f \in C(I)$, then $\Lambda f \in C(I)$ as well. To show this, first note that $\pi \in C(I^\circ)$, $\Pi \in C(I)$, and for $x > 0$ we have $\Pi(x) > 0$. Clearly, then,

$$\Lambda f(x) = \frac{\int_0^x \pi(y) f(y) dy}{\Pi(x)}$$

is continuous at all $x > 0$. Continuity at zero is immediate as for any $\epsilon > 0$, we can choose x such that $|f(y) - f(0)| < \epsilon$ for all $y \leq x$, and so

$$|\Lambda f(0) - \Lambda f(x)| = \left| \int_0^x (f(0) - f(y)) \pi^{(x)}(y) dy \right| \leq \epsilon.$$

Remark 3.3. For $x < 1$, let $\Pi^{(x)}$ be the distribution Π conditioned to $(0, x]$, so that $\Pi^{(x)}$ has density $\pi^{(x)}$ when $x > 0$, and let $\Pi^{(0)} := \delta_0$ and $\Pi^{(1)} := \Pi$. If $\pi_0 = \Pi^{(x)}$ for some $x \in [0, 1]$, then (3.3) is uniquely satisfied by $\pi_0^* = \delta_x$. For $x \in (0, 1)$, this is easily seen via

$$\begin{aligned} (3.4) \quad \int_I f(y) \pi^{(x)}(y) dy &= (\pi_0, f) \\ &= (\pi_0^*, \Lambda f) = \int_I \int_{y \in (0, z]} \pi^{(z)}(y) f(y) dy \pi_0^*(dz) \\ &= \int_I \int_{z \in [y, 1]} \pi^{(z)}(y) \pi_0^*(dz) f(y) dy. \end{aligned}$$

Letting $f(y) = \mathbb{1}(y > x)$ we see π_0^* must be concentrated on $(0, x]$. It also follows that for almost every y satisfying $0 < y \leq x$ we have

$$\pi^{(x)}(y) = \int_{z \in [y, 1]} \pi^{(z)}(y) \pi_0^*(dz) = \int_{z \in [y, x]} \pi^{(z)}(y) \pi_0^*(dz),$$

or, equivalently,

$$\frac{1}{\Pi(x)} = \int_{z \in [y, x]} \frac{\pi_0^*(dz)}{\Pi(z)}.$$

Letting $y \uparrow x$ through such values, it follows that $\pi_0^* = \delta_x$ is the only possible initial distribution for X^* . To show that $\pi_0^* = \delta_x$ satisfies (3.3), note

$$(\delta_x, \Lambda f) = \Lambda f(x) = \int_0^x \pi^{(x)}(y) f(y) dy = (\Pi^{(x)}, f),$$

as desired. For $x = 0$, the argument goes as follows. For uniqueness, if $\pi_0 = \delta_0$, then letting $f(y) = \mathbb{1}(y \in (0, 1])$, the left side of (3.3) equals $f(0) = 0$, and the right side is strictly positive unless $\pi_0^* = \delta_0$. To see that $\pi_0^* = \delta_0$ satisfies (3.3) when $\pi_0 = \Pi^{(0)} = \delta_0$, we compute $(\delta_0, \Lambda f) = \Lambda f(0) = f(0) = (\delta_0, f)$.

3.2. The dual generator. From the definition of strong stationary duality, we derive the form of the dual generator:

Theorem 3.4. *With X as above, assume further that $b \in C^1(I^\circ)$. If X^* is a strong stationary dual of X , then the generator A^* of X^* has the form*

$$(A^* f)(x) = \left(\frac{1}{2} b'(x) - a(x) + b(x) \frac{\pi(x)}{\Pi(x)} \right) f'(x) + \frac{1}{2} b(x) f''(x)$$

for $x \in I^\circ$ and $f \in \mathcal{D}_{A^*}$. Also 0 is an entrance boundary for X^* and 1 is a regular absorbing boundary of X^* .

Proof. Let $f \in \mathcal{D}_A$. Then $Af \in C(I)$ and for $x > 0$ we have

$$\begin{aligned} (\Lambda Af)(x) &= \int_0^x a(y) f'(y) \pi^{(x)}(y) dy + \int_0^x \frac{1}{2} b(y) f''(y) \pi^{(x)}(y) dy \\ &= \frac{1}{\Pi(x)} \left(\int_0^x a(y) f'(y) \pi(y) dy + \int_0^x \frac{1}{2} b(y) f''(y) \pi(y) dy \right). \end{aligned}$$

We know that there exists a nonzero constant C such that $C \cdot \pi(x) = m(x)$, so that $\pi(x) = \frac{1}{Cb(x)s(x)}$. Also, $\frac{d}{dx} \frac{1}{s(x)} = \frac{1}{s(x)} \frac{2a(x)}{b(x)}$. The first term in $(\Lambda Af)(x)$ is then equal to

$$\frac{1}{\Pi(x)} \int_0^x \frac{1}{2C} \frac{2a(y)}{b(y)} \frac{1}{s(y)} f'(y) dy,$$

which by integration by parts equals

$$\frac{1}{\Pi(x)} \frac{1}{2C} \left[\frac{df}{dS}(x) - \frac{df}{dS}^+(0) - \int_0^x \frac{1}{s(y)} f''(y) dy \right].$$

The second term in $(\Lambda Af)(x)$ is equal to

$$\frac{1}{\Pi(x)} \frac{1}{2C} \int_0^x \frac{1}{s(y)} f''(y) dy,$$

and so

$$\begin{aligned} (\Lambda Af)(x) &= \frac{1}{\Pi(x)} \frac{1}{2C} \left[\frac{df}{dS}(x) - \frac{df}{dS}^+(0) \right] \\ &= \frac{1}{2} \frac{b(x)\pi(x)}{\Pi(x)} f'(x) - \frac{1}{\Pi(x)} \frac{1}{2C} \frac{df}{dS}^+(0). \end{aligned}$$

Since 0 is a reflecting boundary of X and $f \in \mathcal{D}_A$, we have $\frac{df}{dS}^+(0) = 0$ and thus

$$(\Lambda Af)(x) = \frac{1}{2} \frac{b(x)\pi(x)}{\Pi(x)} f'(x).$$

Let $g \in \mathcal{D}_{A^*}$. For $x \in (0, 1)$, from equation (2.1) for A^* we can write

$$(A^*g)(x) = a^*(x)g'(x) + \frac{1}{2}b^*(x)g''(x).$$

for some $a^*, b^* \in C(I^\circ)$. If $f \in \mathcal{D}_A$ then by (3.1) we have $\Lambda f \in \mathcal{D}_{A^*}$, and so for $x \in (0, 1)$ we know $(A^*\Lambda f)(x) = a^*(x)(\Lambda f)'(x) + \frac{1}{2}b^*(x)(\Lambda f)''(x)$. Note that $Af \in C(I)$ by assumption and so $\Lambda Af = A^*\Lambda f \in C(I)$ from Remark 3.2. Now

$$\begin{aligned} (\Lambda f)'(x) &= \frac{\Pi(x)\pi(x)f(x) - \pi(x) \int_0^x \pi(y)f(y) dy}{\Pi(x)^2} \\ &= \frac{\pi(x)}{\Pi(x)} [f(x) - (\Lambda f)(x)] \end{aligned}$$

and so

$$\begin{aligned} (\Lambda f)''(x) &= \frac{\Pi(x)\pi'(x) - \pi(x)^2}{\Pi(x)^2} [f(x) - (\Lambda f)(x)] + \frac{\pi(x)}{\Pi(x)} \left\{ f'(x) - \frac{\pi(x)}{\Pi(x)} [f(x) - (\Lambda f)(x)] \right\} \\ &= \left[\frac{\pi'(x)}{\Pi(x)} - \frac{2\pi(x)^2}{\Pi(x)^2} \right] [f(x) - (\Lambda f)(x)] + \frac{\pi(x)}{\Pi(x)} f'(x). \end{aligned}$$

Now by (3.2), $\Lambda A = A^* \Lambda$ as operators on \mathcal{D}_A , which implies that for any $x \in (0, 1)$ and $f \in \mathcal{D}_A$ we have

$$(3.5) \quad \begin{aligned} \frac{1}{2} \frac{b(x)\pi(x)}{\Pi(x)} f'(x) &= \left(a^*(x) \frac{\pi(x)}{\Pi(x)} + \frac{1}{2} b^*(x) \left[\frac{\pi'(x)}{\Pi(x)} - \frac{2\pi(x)^2}{\Pi(x)^2} \right] \right) [f(x) - (\Lambda f)(x)] \\ &\quad + \frac{1}{2} b^*(x) \frac{\pi(x)}{\Pi(x)} f'(x). \end{aligned}$$

For any fixed $x \in I^\circ$, we can choose $f \in \mathcal{D}_A$ so that $f'(x) = 0$ and $f(x) \neq (\Lambda f)(x)$ [e.g., let f be a suitably smooth approximation of $\mathbb{1}(x/3, x/2)$], and for any such f , equation (3.5) yields

$$(3.6) \quad a^*(x) \frac{\pi(x)}{\Pi(x)} + \frac{1}{2} b^*(x) \left[\frac{\pi'(x)}{\Pi(x)} - \frac{2\pi(x)^2}{\Pi(x)^2} \right] = 0.$$

We then find for $f \in \mathcal{D}_A$ and $x \in (0, 1)$ that $(A^* \Lambda f)(x) = \frac{1}{2} \frac{b^*(x)\pi(x)}{\Pi(x)} f'(x)$, and by (3.2) this equals $(\Lambda A f)(x) = \frac{1}{2} \frac{b(x)\pi(x)}{\Pi(x)} f'(x)$. For each x in $(0, 1)$, we can choose an $f \in \mathcal{D}_A$ such that $f'(x) \neq 0$, and using any such f we find that $b^*(x) = b(x)$.

Next, we have from $\pi(x) = \frac{1}{Cb(x)s(x)}$ that

$$\pi'(x) = \frac{-b'(x)s(x) - b(x)s'(x)}{Cb(x)^2 s(x)^2}.$$

Equation (3.6) and $b^* \equiv b$ then yields

$$\begin{aligned} \frac{\pi(x)}{\Pi(x)} a^*(x) &= \frac{1}{2} b(x) \left[\frac{b'(x)s(x) + b(x)s'(x)}{C\Pi(x)b(x)^2 s(x)^2} + \frac{2\pi(x)^2}{\Pi(x)^2} \right] \\ &= \frac{1}{2C\Pi(x)} \left[\frac{b'(x)}{b(x)s(x)} + \frac{s'(x)}{s(x)^2} \right] + b(x) \frac{\pi(x)^2}{\Pi(x)^2} \\ &= \frac{1}{2C\Pi(x)} \left[Cb'(x)\pi(x) - \frac{2a(x)}{s(x)b(x)} \right] + b(x) \frac{\pi(x)^2}{\Pi(x)^2} \\ &= \frac{1}{2} b'(x) \frac{\pi(x)}{\Pi(x)} - a(x) \frac{\pi(x)}{\Pi(x)} + b(x) \frac{\pi(x)^2}{\Pi(x)^2}, \end{aligned}$$

so that $a^*(x) = \frac{1}{2} b'(x) - a(x) + b(x) \frac{\pi(x)}{\Pi(x)}$ on I° , as desired.

To find the boundary behavior of the dual diffusion at 0 and at 1, we calculate the dual scale function and the dual speed measure. First, note that

$$(3.7) \quad \begin{aligned} s^*(x) &= \exp \left[- \int^x \frac{2a^*(y)}{b^*(y)} dy \right] \\ &= \exp \left[- \int^x \frac{b'(y)}{b(y)} dy + \int^x \frac{2a(y)}{b(y)} dy - \int^x \frac{2m(y)}{M(y)} dy \right] \\ &= \frac{1}{b(x)} \frac{1}{s(x)} \frac{1}{M(x)^2} \\ &= \frac{m(x)}{M(x)^2}, \end{aligned}$$

and a scale function for X^* is

$$(3.8) \quad S^*(x) = \frac{-1}{M(x)}.$$

Next, note

$$(3.9) \quad m^*(x) = \frac{1}{b^*(x)s^*(x)} = \frac{M(x)^2}{m(x)b(x)} = M(x)^2 s(x).$$

Now $M(x)$ is continuous on I and $M(0) = 0$, so there is a y such that $M(\zeta) \leq 1$ for all $\zeta \leq y$. For the dual scale measure S^* we then have

$$\begin{aligned} S^*(0, y] &= \int_{(0, y]} s^*(\zeta) d\zeta \\ &= \lim_{z \downarrow 0} \int_z^y \frac{m(\zeta)}{M(\zeta)^2} d\zeta \\ &\geq \lim_{z \downarrow 0} \int_z^y \frac{m(\zeta)}{M(\zeta)} d\zeta \\ &= \lim_{z \downarrow 0} [\log M(y) - \log M(z)] = \infty. \end{aligned}$$

To show that 0 is an entrance boundary for X^* , it now suffices to show that $N^*(0) < \infty$. This is shown via

$$\begin{aligned} N^*(0) &= \lim_{z \downarrow 0} \int_z^x S^*[y, x] dM^*(y) \\ &= \lim_{z \downarrow 0} \int_z^x [S^*(x) - S^*(y)] m^*(y) dy \\ &= \lim_{z \downarrow 0} \int_z^x \left[\frac{-1}{M(x)} - \frac{-1}{M(y)} \right] M(y)^2 s(y) dy \\ &\leq \frac{-1}{M(x)} \liminf_{z \downarrow 0} \int_z^x M(y)^2 s(y) dy + \limsup_{z \downarrow 0} \int_z^x M(y) s(y) dy. \end{aligned}$$

It now clearly suffices to prove $\int_0^x M(y) s(y) dy < \infty$, which follows from the following calculation:

$$\int_0^x M(y) s(y) dy = \int_0^x M(y) dS(y) = \int_0^x S[y, x] dM(y) =: N(0) < \infty,$$

where we used the fact that 0 is a reflecting boundary for X to derive the final inequality.

To prove that 1 is a regular absorbing boundary for X^* , we use Proposition 3.5 below. From that proposition, for any $f \in C[0, 1]$ and $x \in [0, 1]$, we have $(\Lambda T_t f)(x) = (T_t^* \Lambda f)(x)$. When $x > 0$, we then have

$$\begin{aligned} \int_0^1 \left[\int_0^x \pi^{(x)}(y) p_t(y, z) dy \right] f(z) dz &= \int_{[0, 1]} \int_0^y \pi^{(y)}(z) f(z) dz P_x^*(X_t^* \in dy) \\ &= \int_0^1 \left[\int_{[z, 1]} \pi^{(y)}(z) P_x^*(X_t^* \in dy) \right] f(z) dz. \end{aligned}$$

In particular, letting $x = 1$ we find

$$\int_0^1 \pi(z) f(z) dz = \int_0^1 \left[\int_{[z, 1]} \pi^{(y)}(z) P_1^*(X_t^* \in dy) \right] f(z) dz.$$

Since this holds for all $f \in C[0, 1]$, and since both $\pi(z)$ and the expression in square brackets on the right are continuous functions of $z \in (0, 1]$, it follows, for all

$z \in (0, 1]$, that

$$\pi(z) = \int_{[z, 1]} \frac{\pi(z)}{\Pi(y)} P_1^*(X_t^* \in dy),$$

and hence $\int_{[z, 1]} \frac{1}{\Pi(y)} P_1^*(X_t^* \in dy) = 1$. It now follows that $P_1^*(X_t^* = 1) = 1$ and hence that the boundary 1 is either regular absorbing or exit. To show that the boundary is absorbing, it suffices to show that $N^*(1) < \infty$. Indeed, for fixed x in I° we have [using (3.8)–(3.9)] that

$$\begin{aligned} N^*(1) &= \int_{[x, 1)} [S^*(y) - S^*(x)] m^*(y) dy = \int_{[x, 1)} \left[\frac{1}{M(x)} - \frac{1}{M(y)} \right] s(y) M^2(y) dy \\ &= \int_{[x, 1)} s(y) \frac{M^2(y)}{M(x)} dy - \int_{[x, 1)} s(y) M(y) dy \\ &\leq \frac{M(1)}{M(x)} \int_{[x, 1)} s(y) M(y) dy - \int_{[x, 1)} s(y) M(y) dy < \infty \end{aligned}$$

where the finiteness holds since 1 is reflecting for X [hence $\Sigma(1) < \infty$] and $M(\cdot)$ is increasing and bounded on I° . \square

Proposition 3.5. *Let X^* be a strong stationary dual of X , and let the one-parameter Markov semigroups of operators for X^* and X be (T_t^*) and (T_t) respectively. Then for all t we have $\Lambda T_t = T_t^* \Lambda$ as operators on $C[0, 1]$.*

Proof. For all λ we have $\Lambda(\lambda I - A) = (\lambda I - A^*)\Lambda$ and so the resolvent operators satisfy $\Lambda R_\lambda = R_\lambda^* \Lambda$. For $f \in C[0, 1]$ and $x \in [0, 1]$, note that

$$(R_\lambda^* \Lambda f)(x) = \int_0^\infty e^{-\lambda t} (T_t^* \Lambda f)(x) dt$$

and that

$$\begin{aligned} (\Lambda R_\lambda f)(x) &= \int_0^x \pi^{(x)}(y) (R_\lambda f)(y) dy \\ &= \int_0^x \int_0^\infty \pi^{(x)}(y) e^{-\lambda t} (T_t f)(y) dt dy \\ &= \int_0^\infty e^{-\lambda t} (\Lambda T_t f)(x) dt. \end{aligned}$$

Now, by the uniqueness of Laplace transforms of real valued functions, we have $(\Lambda T_t f)(x) = (T_t^* \Lambda f)(x)$ for all t , as desired. \square

Remark 3.6. From Proposition 3.5, we have that $\Lambda T_t = T_t^* \Lambda$ as operators on $C[0, 1]$ which implies that the equality also holds as operators on $F[0, 1]$.

The choice of 0 and 1 as instantaneously reflecting boundaries was done to streamline exposition. However, we can establish analogues of Theorem 3.4 for more general boundary behaviors of X . If 0 and 1 are entrance boundaries for X , then the domain of A is

$$\mathcal{D}_A = \{f \in C(I) \cap C^2(I^\circ) \mid Af \in C(I)\}.$$

If 0 (resp., 1) is made reflecting then we impose the extra condition that $\frac{df}{ds}^+(0) = 0$ [resp., $\frac{df}{ds}^-(1) = 0$] for functions $f \in \mathcal{D}_A$. In the proof of Theorem 3.4, only the

following properties of the boundary at 0 were needed:

$$\frac{df}{dS}^+(0) = 0 \text{ for } f \in C(I), \quad N(0) < \infty,$$

and these properties also hold if 0 is an entrance boundary. Absorption of X^* at 1 is proven completely analogously to the reflecting case. If 1 is an entrance boundary for X , then 1 is an exit boundary for X^* since

$$\begin{aligned} N^*(1) &= \int_{[x,1)} [S^*(y) - S^*(x)] m^*(y) dy = \int_{[x,1)} \left[\frac{1}{M(x)} - \frac{1}{M(y)} \right] s(y) M^2(y) dy \\ &= \int_{[x,1)} s(y) \left[\frac{M^2(y)}{M(x)} - M(y) \right] dy \\ &\geq \int_x^1 s(y) [M(y) - M(x)] dy = \Sigma(1) = \infty \end{aligned}$$

and (twice utilizing integration by parts)

$$\begin{aligned} \Sigma^*(1) &= \int_x^1 [S^*(1) - S^*(y)] m^*(y) dy = \int_x^1 \left[\frac{1}{M(y)} - \frac{1}{M(1)} \right] s(y) M^2(y) dy \\ &= \int_x^1 s(y) \left[M(y) - \frac{M^2(y)}{M(1)} \right] dy \\ &\leq \int_x^1 s(y) [M(1) - M(y)] dy = N(1) < \infty. \end{aligned}$$

We thus arrive at the following generalization of Theorem 3.4.

Theorem 3.7. *Let X be a regular diffusion on I , and assume that each of the boundary points of I is either reflecting or entrance. Assume further that $b \in C^1(I^\circ)$. If X^* is a strong stationary dual of X , then the generator A^* of X^* has the form*

$$(A^*f)(x) = \left(\frac{1}{2}b'(x) - a(x) + b(x) \frac{\pi(x)}{\Pi(x)} \right) f'(x) + \frac{1}{2}b(x)f''(x)$$

for $x \in I^\circ$ and $f \in \mathcal{D}_{A^*}$. Also 0 is an entrance boundary for X^* . If 1 is a reflecting boundary of X , then 1 is a regular absorbing boundary of X^* . If 1 is an entrance boundary of X , then 1 is an exit boundary of X^* .

Example 3.8. For $\alpha \geq 0$, a diffusion X on $[0, 1]$ is said to be a Bessel process with parameter α [written $\text{Bes}(\alpha)$], reflected at 1, if the generator of X has the form

$$A = \frac{1}{2} \frac{d^2}{dx^2} + \frac{\alpha - 1}{2x} \frac{d}{dx},$$

and if for $f \in \mathcal{D}_A$ we have $\frac{df}{dS}^-(1) = 0$. The behavior at the boundary 0 is determined by the value of α . For $0 < \alpha < 2$, the boundary 0 is a regular reflecting boundary, and for $\alpha \geq 2$ the boundary 0 is an entrance boundary. For our discussion of duality, we do not consider the case $\alpha = 0$, for which 0 is an absorbing boundary. For $\alpha > 0$, a simple application of Theorem 3.7 gives that if X is a $\text{Bes}(\alpha)$ process on $[0, 1]$ with instantaneously reflecting behavior at 1 begun in $\pi^{(x)}$, then X^* is a $\text{Bes}(\alpha+2)$ process begun in δ_x absorbed at 1. In particular, the dual of reflecting Brownian motion, i.e., the $\text{Bes}(1)$ process reflected at 1, is the $\text{Bes}(3)$ process reflected at 1. For an extensive background treatment of Bessel processes, see [20, Chapter 4.3] or [27, Chapter V–VI].

Example 3.9. For a second example, we turn to the Wright–Fisher gene frequency model from population genetics. The Wright–Fisher diffusion X is a diffusion on $[0, 1]$ with generator of the form

$$A = \frac{1}{2}x(1-x)\frac{d^2}{dx^2} + [\alpha(1-x) - \beta x]\frac{d}{dx}.$$

The behavior at the boundaries is determined by the values of α and β . We have that

$$0 \text{ is a(n) } \begin{cases} \text{entrance boundary if } \alpha \geq 1/2, \\ \text{reflecting regular boundary if } 0 < \alpha < 1/2, \\ \text{exit boundary if } \alpha = 0, \end{cases}$$

and

$$1 \text{ is a(n) } \begin{cases} \text{entrance boundary if } \beta \geq 1/2, \\ \text{reflecting regular boundary if } 0 < \beta < 1/2, \\ \text{exit boundary if } \beta = 0. \end{cases}$$

If X is a Wright–Fisher diffusion with $\alpha = 1/2$ and $\beta > 0$, then from Theorem 3.7 we have that the strong stationary dual of X is a Wright–Fisher diffusion with $\alpha^* = \alpha + (1/2)$ and $\beta^* = 0$. For an extensive background on the Wright–Fisher model and its many applications, see [18, Section 15.8] or [8, Chapter 10].

Not surprisingly, we can also recover a partial converse to Proposition 3.5.

Lemma 3.10. *Let X and X^* be diffusions on $[0, 1]$ and let 0 and 1 be either instantaneously reflecting or entrance boundaries for X . Then an intertwining*

$$(3.10) \quad \Lambda T_t = T_t^* \Lambda \text{ (for all } t \geq 0)$$

of the one-parameter semigroups by the link Λ together with the initial condition (3.3) implies that X^ is a strong stationary dual of X .*

Proof. Let $f \in \mathcal{D}_A$. Then by definition $(Af)(x) = \lim_{t \downarrow 0} \frac{T_t f(x) - f(x)}{t}$. We have that $(Af)(x) = \frac{1}{2}b(x)f''(x) + a(x)f'(x)$ is continuous in x and so the convergence of $\frac{T_t f(x) - f(x)}{t}$ to $(Af)(x)$ is uniform (Theorem 7.7.3 in [21]). Now

$$\begin{aligned} (\Lambda Af)(x) &= \int_0^x \pi^{(x)}(y) \left[\lim_{t \downarrow 0} \frac{T_t f(y) - f(y)}{t} \right] dy \\ &= \lim_{t \downarrow 0} \int_0^x \pi^{(x)}(y) \frac{T_t f(y) - f(y)}{t} dy \\ &= \lim_{t \downarrow 0} \frac{(\Lambda T_t f)(x) - (\Lambda f)(x)}{t} \\ &= \lim_{t \downarrow 0} \frac{(T_t^* \Lambda f)(x) - (\Lambda f)(x)}{t} \\ &= (A^* \Lambda f)(x), \end{aligned}$$

where the last limit's existence is guaranteed by that of the first. This gives both that $\Lambda|_{\mathcal{D}_A} \subset \mathcal{D}_{A^*}$, and that on \mathcal{D}_A we have $\Lambda A = A^* \Lambda$ as desired. \square

Remark 3.11. Intertwinings of Markov semigroups have been well studied, appearing for example in [7], [26], etc. In the context of (3.10), the transition operator Λ is the following Markov kernel from $[0, 1]$ to $[0, 1]$: For $x \in [0, 1]$ and $A \in \mathcal{B}$ we have

$$\Lambda(x, A) = \Pi^{(x)}(A).$$

Remark 3.12. If (3.10) holds, then

$$(U_t \pi_0, f) = (\pi_0, T_t f) = (\pi_0^*, \Lambda T_t f) = (\pi_0^*, T_t^* \Lambda f) = (U_t^* \pi_0^*, \Lambda f),$$

mirroring the corresponding result that algebraic duality via link L of Markov chains yields $\pi_t = \pi_t^* L$.

4. APPROXIMATING DUALITY VIA MARKOV CHAINS

The purpose of the present section is twofold. Presently suppressing all details (which will be spelled out in full detail later in the section), we will show that a suitably defined sequence of Markov chains X^Δ and their corresponding strong stationary duals \hat{X}^Δ , as defined in [4], converge respectively to our primal diffusion $Y = S(X)$ (in natural scale) and its strong stationary dual Y^* . By establishing the newly defined diffusion strong stationary dual as a limit of an appropriately defined sequence of classical Markov chain strong stationary duals, we ground our definition and our present work in the classical theory.

In addition to tethering our duality to the classical theory, this has a number of interesting consequences. For example, we believe one of the great triumphs of strong stationary duality was its application in the perfect sampling algorithms of [10] and [13]. Via the work in the present section, for our primal diffusion Y we could approximately sample perfectly from Π_Y by using the theory of [10] to perfectly sample from the stationary distributions of the approximating sequence of chains. We could also use our approximating sequence of chains to study cut-off type behaviors of the dual hitting times of state $S(1)$, and hence of the primal diffusion's separation distance from stationarity. We are also able to recover the dual-hitting-time/primal-mixing-time duality of the classical Markov chain theory in the diffusion setting by passing to appropriate limits; see Section 5 for full details.

This section is laid out as follows: First assuming instantaneously reflecting boundaries for our primal diffusion Y , in Sections 4.1–4.2 we explicitly spell out the one-dimensional convergence of our primal and dual sequences of Markov chains to the corresponding primal and dual diffusions. In Section 4.3, we prove the corresponding convergence theorems in the case when our primal diffusion has entrance boundaries at 0 and/or 1.

4.1. Primal convergence. Let $D_I[0, \infty)$ be the space of cadlag functions from $[0, \infty)$ into I . We can equip $D_I[0, \infty)$ with a metric d defined by

$$d(x, y) = \inf_{\lambda \in B} \left[\left(\sup_{s > t \geq 0} \left| \log \frac{\lambda(s) - \lambda(t)}{s - t} \right| \right) \vee \int_0^\infty e^{-u} d(x, y, \lambda, u) du \right]$$

where B is the set of strictly increasing Lipschitz continuous functions from $[0, \infty)$ to $[0, \infty)$ with the additional property that

$$\lambda \in B \text{ implies } \sup_{s > t \geq 0} \left| \log \frac{\lambda(s) - \lambda(t)}{s - t} \right| < \infty,$$

and

$$d(x, y, \lambda, u) := \sup_{t \geq 0} (|x(t \wedge u) - y(\lambda(t) \wedge u)| \wedge 1).$$

The topology induced by d is known as the *Skorohod topology*, and under this topology $D_I[0, \infty)$ is both complete and separable (as I is both complete and separable). For more background on $D_I[0, \infty)$, see [8, Sections 3.5–3.10] or [3, Chapters 2–3].

We will consider stochastic processes with sample paths in $D_I[0, \infty)$ as $D_I[0, \infty)$ -valued random variables and we will say that $X_n \Rightarrow X$ if we have convergence in law of the corresponding $D_I[0, \infty)$ -valued random variables. Note that $X_n \Rightarrow X$ implies convergence of the associated finite-dimensional distributions of X_n to those of X (see [8, Theorem 3.7.8]), i.e., for all $\{t_1, \dots, t_m\} \subset \{t \geq 0 \mid \mathbb{P}(X(t) = X(t-)) = 1\}$ we have

$$(X_n(t_1), \dots, X_n(t_m)) \Rightarrow (X(t_1), \dots, X(t_m)).$$

As in Section 2, let X be a regular diffusion on I with instantaneous reflection at the boundaries of I and scale function $S \equiv S_X$. To ease exposition, we will consider $Y = S_X(X)$, a regular diffusion in natural scale on $\mathcal{S} = [S_X(0), S_X(1)]$, and assume S_Y has been scaled to make $s_Y \equiv 1$. The speed function of Y is $M_Y = M_X \circ S_X^{-1} : \mathcal{S}^\circ \rightarrow \mathbb{R}$ (where M_X is the speed function of X). As with X , define the speed measure of Y as the nonnegative measure on \mathcal{S}° , denoted $M_Y(\cdot)$, defined via $M_Y(x, y] = M_Y(y) - M_Y(x)$.

As X is regular and $\mathbb{P}_x(X_t = 0) = 0$ for all $t > 0$ and $x \in I$, it follows that Y is regular and $\mathbb{P}_x(X_t = 0) = \mathbb{P}_{S(x)}(Y_t = S(0)) = 0$ for all $t > 0$ and $S(x) \in \mathcal{S}$. Therefore $S(0)$ is an instantaneously reflecting boundary for Y . Analogous results hold at $S(1)$, and it follows that $S(1)$ is an instantaneously reflecting boundary for Y and $M_Y(\{S(1)\}) = 0$.

The generator of Y can be expressed as $(A_Y f)(y) = \frac{1}{2} b_Y(y) f''(y)$ with $b_Y(y) = b_X(x) s_X^2(x)$ where $y = S_X(x)$. Note that $M_Y(\mathcal{S}^\circ) = M_X(I^\circ) < \infty$ and so there exists a unique invariant measure for Y which we will denote Π_Y . Observe

$$\begin{aligned} M_Y((c, d]) &= M_Y(d) - M_Y(c) = M_X(S^{-1}(d)) - M_X(S^{-1}(c)) \\ &= \int_{S^{-1}(c)}^{S^{-1}(d)} m_X(z) dz = \int_c^d \frac{m_X(S_X^{-1}(w))}{s_X(S_X^{-1}(w))} dw \end{aligned}$$

so that M_Y (resp., Π_Y) has density $m_Y(y) = m_X(S_X^{-1}(y))/s_X(S_X^{-1}(y))$ (resp., density $\pi_Y = \alpha m_Y$ for some constant α). On \mathcal{S}° we have $m_Y = b_Y^{-1}$ and so $\pi_Y b_Y$ is constant on \mathcal{S}° . Assume that b_Y can be extended to a function in $C(\mathcal{S})$, so that $b_Y(S(0))$ and $b_Y(S(1))$ are well defined, and assume that $\lim_{y \rightarrow z} \pi_Y(y) b_Y(y) = \alpha$ for $z \in \{S(0), S(1)\}$.

For the remainder of the section, we shall be working with the diffusion Y rather than X , and so we will drop the Y subscript from b_Y , π_Y , Π_Y , etc.

Let $\Delta > 0$ be such that $S(1) - S(0) = n^\Delta \Delta$ for some integer n^Δ . As in [2, Chapter 6], define a birth-and-death transition matrix P^Δ on state space

$$\mathcal{S}^\Delta := \{S(0), S(0) + \Delta, S(0) + 2\Delta, \dots, S(1) - \Delta, S(1)\}$$

by setting (for ease of notation, we write i for $S(0) + i\Delta$ here):

$$P^\Delta(i, i+1) = P^\Delta(i, i-1) := \frac{b(i)h}{2\Delta^2} \quad \text{for } 0 < i < n^\Delta \text{ and}$$

$$P^\Delta(0, 1) := \frac{b(0)h}{\Delta^2}, \quad P^\Delta(n^\Delta, n^\Delta - 1) := \frac{b(n^\Delta)h}{\Delta^2};$$

here

$$(4.1) \quad h \equiv h_\Delta := \frac{\Delta^2}{2 \sup_y b(y)}$$

is chosen to make P^Δ monotone.

Note that for $i \in \{1, \dots, n^\Delta - 1\}$ we have

$$\pi(i)P^\Delta(i, i+1) = \pi(i+1)P^\Delta(i+1, i),$$

and at the boundaries we have

$$\pi(0)P^\Delta(0, 1) = 2\pi(1)P^\Delta(1, 0), \quad \pi(n^\Delta)P^\Delta(n^\Delta, n^\Delta-1) = 2\pi(n^\Delta-1)P^\Delta(n^\Delta-1, n^\Delta).$$

It follows that there exists a constant C^Δ such that

$$(4.2) \quad \pi^\Delta(i) = \begin{cases} C^\Delta \pi(i), & i = 1, \dots, n^\Delta - 1; \\ C^\Delta \pi(i)/2, & i = 0, n^\Delta \end{cases}$$

is the unique invariant probability distribution for P^Δ .

Let π_0^Δ be a probability measure on \mathcal{S}^Δ , and let P^Δ be the transition matrix for a discrete-time birth-and-death chain X^Δ , begun in π_0^Δ , on state space \mathcal{S}^Δ [we write $X^\Delta \sim (\pi_0^\Delta, P^\Delta)$ as shorthand].

Theorem 4.1. *Assume there exists a constant $\delta > 0$ such that $b \geq \delta$ everywhere and that we can continuously extend b to the boundaries of \mathcal{S} . Consider a sequence of values $\Delta \downarrow 0$ such that for each Δ we have $S(1) - S(0) = n^\Delta \Delta$ for some integer n^Δ . Define the continuous-time stochastic process Y^Δ by setting $Y_t^\Delta := X_{\lfloor t/h_\Delta \rfloor}^\Delta$ for $t \geq 0$. If $Y_0^\Delta \Rightarrow Y_0$, then $Y^\Delta \Rightarrow Y$.*

Our main proof tool will be the following theorem, adapted from [8, Corollary 4.8.9 and Theorem 1.6.5]:

Theorem 4.2. *Let A be the generator of a regular diffusion process Y with state space \mathcal{S} . Assume $h_\Delta > 0$ converges to 0 as $\Delta \downarrow 0$. Let $Y_t^\Delta := X_{\lfloor t/h_\Delta \rfloor}^\Delta$ where $X^\Delta \sim (\pi_0^\Delta, P^\Delta)$ is a Markov chain with some metric state space $\mathcal{S}^\Delta \subset \mathcal{S}$, and assume $Y_0^\Delta \Rightarrow Y_0$. Define $T^\Delta : B(\mathcal{S}^\Delta) \rightarrow B(\mathcal{S}^\Delta)$ via*

$$T^\Delta f(x) = \mathbb{E}_x f(X_1^\Delta)$$

for f in the space $B(\mathcal{S}^\Delta)$ of real-valued bounded measurable functions on \mathcal{S}^Δ . Define $A^\Delta := h_\Delta^{-1}(T^\Delta - I)$. Suppose that $C(\mathcal{S})$ is convergence determining and that there is an algebra $B \subset C(\mathcal{S})$ that strongly separates points. Let $\rho_\Delta : C(\mathcal{S}) \rightarrow B(\mathcal{S}^\Delta)$ be defined via $\rho_\Delta f(\cdot) = f|_{\mathcal{S}^\Delta}(\cdot)$. If

$$(4.3) \quad \lim_{\Delta \rightarrow 0} \sup_{y \in \mathcal{S}^\Delta} |(A^\Delta \rho_\Delta f)(y) - (Af)(y)| = 0$$

for all $f \in \mathcal{D}_A$, then $Y^\Delta \Rightarrow Y$.

The adaptation of Theorem 4.2 from [8, Corollary 4.8.9 and Theorem 1.6.5] is spelled out explicitly in Appendix A, as the notation between [8] and the present section differs considerably.

Proof of Theorem 4.1. Clearly $C(\mathcal{S})$ is convergence determining, and by considering suitably smooth uniform approximations to the indicator function of $\{x\}$ in \mathcal{D}_A for each $x \in \mathcal{S}$, it follows that $\mathcal{D}_A \subset C(\mathcal{S})$ is an algebra that strongly separates points. Let $f \in \mathcal{D}_A$, so that $(Af)(y) = \frac{1}{2}b(y)f''(y)$. Using

$$\mathcal{D}_A = \{f \in C(\mathcal{S}) \cap C^2(\mathcal{S}^\circ) \mid Af \in C(\mathcal{S}), f'(S(0)+) = f'(S(1)-) = 0\},$$

we find that $bf'' \in C(\mathcal{S})$. As $b(y) \geq \delta > 0$ for all $y \in \mathcal{S}$, we have $1/b \in C(\mathcal{S})$ and therefore $f'' \in C(\mathcal{S})$. It follows that as $\Delta \downarrow 0$, uniformly for $y \in \mathcal{S}^\Delta \setminus \{S(0), S(1)\}$ we have

$$h^{-1}((T^\Delta - I)f)(y) = \frac{1}{2}b(y) \frac{f(y + \Delta) - 2f(y) + f(y - \Delta)}{\Delta^2} \rightarrow \frac{1}{2}b(y)f''(y).$$

Likewise,

$$\begin{aligned} h^{-1}((T^\Delta - I)f)(S(0)) &= b(S(0)) \frac{f(S(0) + \Delta) - f(S(0))}{\Delta^2} \rightarrow \frac{1}{2}b(S(0))f''(S(0)), \\ h^{-1}((T^\Delta - I)f)(S(1)) &= b(S(1)) \frac{f(S(1) - \Delta) - f(S(1))}{\Delta^2} \rightarrow \frac{1}{2}b(S(1))f''(S(1)). \end{aligned}$$

Therefore

$$\sup_{y \in \mathcal{S}^\Delta} |(A^\Delta \rho_\Delta f)(y) - (Af)(y)| \rightarrow 0,$$

establishing (B.1). The result follows. \square

Remark 4.3. For $x \in (S(0), S(1))$, let $i_{\Delta,x} := \lfloor [x - S(0)]/\Delta \rfloor$, and denote the invariant measure π^Δ truncated to $\{S(0), S(0) + \Delta, \dots, S(0) + i_{\Delta,x}\Delta\}$ by $\pi^{\Delta, i_{\Delta,x}}$. If Y is begun with density $\pi^{(x)}$, then for $y \in (S(0) + k\Delta, S(0) + (k+1)\Delta)$ with $0 \leq k < i_{\Delta,x}$ we have

$$\mathbb{P}^\Delta(Y_0^\Delta \leq y) = \frac{\Delta \sum_{j=0}^k \pi^\Delta(S(0) + j\Delta)}{\Delta \sum_{j=0}^{i_{\Delta,x}} \pi^\Delta(S(0) + j\Delta)} \rightarrow \frac{\int_{S(0)}^y \pi(z) dz}{\int_{S(0)}^x \pi(z) dz} = \mathbb{P}(Y_0 \leq y).$$

If X^Δ is begun in $\pi^{\Delta, i_{\Delta,x}}$, it follows that $X_0^\Delta = Y_0^\Delta \Rightarrow Y_0$. If instead Y is begun deterministically at $S(0)$, then letting $X_0^\Delta = S(0)$ for all Δ again gives $X_0^\Delta = Y_0^\Delta \Rightarrow Y_0$.

Remark 4.4. For the sequence of birth-and-death chains (π_0^Δ, P^Δ) , where either $\pi_0^\Delta = \delta_{S(0)}$ for each Δ or $x \in (S(0), S(1))$ is given and $\pi_0^\Delta = \pi^{\Delta, i_{\Delta,x}}$ for each Δ , we are guaranteed the existence of a sequence of birth-and-death strong stationary dual chains by [4, eqs. (4.16a)–(4.16b)] because of the following two observations.

(a) P^Δ is monotone. Indeed, for $i = 0, \dots, n^\Delta - 1$ we easily see

$$P^\Delta(i, i+1) + P^\Delta(i+1, i) \leq 1.$$

(b) The ratio $\pi_0^\Delta / \pi^\Delta$ of probability mass functions (initial to stationary) is non-increasing.

4.2. Dual convergence. As in [4], construct on the same probability space as for X^Δ a strong stationary dual $\hat{X}^\Delta \sim (\hat{\pi}_0^\Delta, \hat{P}^\Delta)$ of X^Δ using the link Λ of truncated stationary distributions [here, for ease of notation, i is again used as shorthand for $S(0) + i\Delta$]:

$$\Lambda^\Delta(i, j) := \mathbb{1}\{j \leq i\} \frac{\pi^\Delta(j)}{H^\Delta(i)};$$

we have used the shorthand $H^\Delta(i) := \sum_{j=0}^i \pi^\Delta(j)$. Note that \hat{X}^Δ is also a birth-and-death chain on \mathcal{S}^Δ . Assume either that $X_0^\Delta = S(0)$ for every Δ so that $\hat{X}_0^\Delta = S(0)$ for every Δ as well, or that $X_0^\Delta \sim \pi^{\Delta, i_{\Delta,x}}$ for every Δ so that $\hat{X}_0^\Delta = S(0) + i_{\Delta,x}\Delta$. The one-step transition matrix \hat{P}^Δ for \hat{X}^Δ is given by

$$(4.4) \quad \hat{P}^\Delta(i, i-1) = \frac{H^\Delta(i-1)}{H^\Delta(i)} \frac{b(i)h}{2\Delta^2} = \frac{b(i)h}{2\Delta^2} - \frac{h \cdot \alpha \cdot C^\Delta}{H^\Delta(i)2\Delta^2} \quad \text{for } 0 < i < n^\Delta,$$

$$(4.5) \quad \hat{P}^\Delta(i, i+1) = \frac{H^\Delta(i+1)}{H^\Delta(i)} \frac{b(i+1)h}{2\Delta^2} = \frac{h \cdot \alpha \cdot C^\Delta}{H^\Delta(i)2\Delta^2} + \frac{b(i+1)h}{2\Delta^2} \quad \text{for } 0 < i < n^\Delta,$$

$$(4.6) \quad \hat{P}^\Delta(0, 1) = \frac{H^\Delta(1)}{H^\Delta(0)} \frac{b(1)h}{\Delta^2},$$

$$(4.7) \quad \hat{P}^\Delta(n_\Delta, n_\Delta) = 1,$$

with $\hat{P}^\Delta(i, i)$ having values for $0 \leq i < n^\Delta$ so that the rows of \hat{P}^Δ sum to unity. We next show the following theorem:

Theorem 4.5. *Assume that $b \in C^2(S(0), S(1)]$, and $\pi > 0$ on \mathcal{S} . Define the continuous-time processes $(\hat{Y}_t^\Delta) := (\hat{X}_{[t/h]}^\Delta)$, and assume $\hat{Y}_0^\Delta \Rightarrow Y_0^*$. Then*

$$\hat{Y}^\Delta \Rightarrow Y^*,$$

where Y^* is a SSD of Y in the sense of Definition 3.1.

We will prove Theorem 4.5 after a series of preliminary results. We begin by putting Y^* into natural scale, i.e., consider the diffusion $Z^* = S^*(Y^*) = -1/M(Y^*)$ on state space $\mathcal{S}^* = (S^*(S(0)), S^*(S(1))]$. Note that the infinitesimal parameters of Z^* are given on $(-\infty, S^*(S(1)))$ [recalling (3.7)] by

$$a_{Z^*} \equiv 0, \quad b_{Z^*}(S^*(y)) = b(y)s^*(y)^2 = \frac{b(y)m^2(y)}{M^4(y)} = \alpha^2 \frac{b(y)\pi^2(y)}{\Pi^4(y)}$$

(recall α is the constant such that $\pi = \alpha \cdot m = \alpha/b$). Note also that under the assumptions of Theorem 4.5, we have $b \in C^2(S(0), S(1)]$ [and so $\pi(\cdot) \propto b(\cdot)$ implies $\pi \in C^2(S(0), S(1))$], $\pi > 0$ on \mathcal{S} . Note also that

$$\begin{aligned} (S^*)'(i) &= \alpha \frac{\pi(i)}{\Pi^2(i)}, \\ (S^*)''(i) &= \alpha \frac{\Pi^2(i)\pi'(i) - 2\pi^2(i)\Pi(i)}{\Pi^4(i)}, \\ (S^*)'''(i) &= \alpha \frac{\Pi^2(i)\pi''(i) - 2\pi'(i)\Pi(i)\pi(i)}{\Pi^4(i)} - \alpha \frac{4\Pi^3(i)\pi(i)\pi'(i) - 6\pi^3(i)\Pi^2(i)}{\Pi^6(i)}, \end{aligned}$$

which implies that $S^* \in C^3[-S(0) + i_0\Delta, S(1)]$ for any $i_0 > 0$.

Define

$$\hat{Z}_t^\Delta := S^*(\hat{X}_t^\Delta),$$

and note that this is a birth and death chain on state space

$$\mathcal{S}^{*,\Delta} := \{S^*(S(0)), S^*(S(0) + \Delta), \dots, S^*(S(1) - \Delta), S^*(S(1))\}.$$

Define

$$(4.8) \quad a_{\hat{Z}^\Delta}(x) := \frac{1}{h_\Delta} \mathbb{E}_x \left(\hat{Z}_1^\Delta - x \right),$$

where we require that $x = S^*(S(0) + i\Delta)$ for some nonnegative integer i .

Proposition 4.6. *With the assumptions of Theorem 4.5, letting $R < \infty$ be fixed, it follows that*

$$\lim_{\Delta \downarrow 0} \sup_{|x| < R} |a_{\hat{Z}^\Delta}(x)| = 0.$$

Proof. Abbreviate $S(0) + i\Delta$ as i , and then x is of the form $x = S^*(i)$. Let

$$A := \frac{S^*(i+1) - 2S^*(i) + S^*(i-1)}{\Delta^2} \frac{b(i)}{2}; \quad B := \frac{S^*(i+1) - S^*(i)}{\Delta} \frac{b(i+1) - b(i)}{2\Delta};$$

$$C := \frac{\alpha \cdot C^\Delta}{\Delta H^\Delta(i)} \frac{S^*(i+1) - S^*(i-1)}{2\Delta}.$$

Then (4.4)–(4.5) allow us to rewrite (4.8) as

$$a_{\widehat{Z}_\Delta}(x) = A + B + C$$

for $x \neq S^*(S(1))$. For all $|x| = S^*(i) < R$ we have $0 < \delta < i\Delta < \gamma < \infty$ uniformly in Δ for some γ and δ . A Taylor expansion of $S^*(\cdot)$ combined with $S^* \in C^3[\delta, S(1)]$ gives

$$\lim_{\Delta \downarrow 0} \sup_{|x| < R, x \neq S^*(S(1))} \left| A - \alpha \frac{\Pi^2(i)\pi'(i) - 2\pi^2(i)\Pi(i)}{\Pi^4(i)} \frac{b(i)}{2} \right| = 0,$$

or equivalently

$$\lim_{\Delta \downarrow 0} \sup_{|x| < R, x \neq S^*(S(1))} \left| A - \frac{\alpha^2}{2} \left(\frac{\pi'(i)}{\pi(i)\Pi^2(i)} - \frac{2\pi(i)}{\Pi^3(i)} \right) \right| = 0;$$

and

$$\lim_{\Delta \downarrow 0} \sup_{|x| < R, x \neq S^*(S(1))} \left| B - \alpha \frac{\pi(i)}{\Pi^2(i)} \frac{b'(i)}{2} \right| = 0,$$

or equivalently

$$\lim_{\Delta \downarrow 0} \sup_{|x| < R, x \neq S^*(S(1))} \left| B + \alpha^2 \frac{\pi'(i)}{2\Pi^2(i)\pi(i)} \right| = 0.$$

Next, for C note that $(C^\Delta)^{-1}\Delta H^\Delta(i) > 0$ if $i\Delta > \delta$ (which is satisfied uniformly by all $|x| < R$). The function $\pi \circ S^*(\cdot)$ is uniformly continuous on $[-R, R]$, and, since the Riemann sum of a uniformly continuous function converges uniformly to the corresponding Riemann integral, we have for the sup-norm $\|\cdot\|_\infty$ on $[-R, R]$ that

$$\lim_{\Delta \downarrow 0} \|(C^\Delta)^{-1}\Delta H^\Delta - \Pi\|_\infty = 0.$$

By regularity of the primal diffusion Y , we have $\Pi(i) > 0$ for $i\Delta > \delta > 0$, and therefore for such i we have $(C^\Delta)^{-1}\Delta H^\Delta(i)\Pi(i) > 0$. Note that $(C^\Delta)^{-1}\Delta H^\Delta(\cdot)\Pi(\cdot)$ is a bounded increasing function in i . All of this leads to

$$\lim_{\Delta \downarrow 0} \sup_{|x| < R, x \neq S^*(S(1))} \left| \frac{C^\Delta}{\Delta H^\Delta(i)} - \frac{1}{\Pi(i)} \right| =$$

$$\lim_{\Delta \downarrow 0} \sup_{|x| < R, x \neq S^*(S(1))} \left| \frac{C^\Delta}{\Delta H^\Delta(i)\Pi(i)} \right| \cdot \left| \Pi(i) - \frac{\Delta H^\Delta(i)}{C^\Delta} \right| = 0.$$

Therefore

$$\lim_{\Delta \downarrow 0} \sup_{|x| < R, x \neq S^*(S(1))} \left| C - \frac{\alpha^2 \pi(i)}{\Pi(i)^3} \right| = 0.$$

Combining our results for A , B , and C with the observations that $a_{\widehat{Z}_\Delta}(S^*(S(1))) = 0$, we find

$$\lim_{\Delta \downarrow 0} \sup_{|x| < R} |a_{\widehat{Z}_\Delta}(x)| = 0,$$

as desired. \square

Next, define

$$(4.9) \quad b_{\widehat{Z}_\Delta}(x) := \frac{1}{h_\Delta} \mathbb{E}_x \left(\widehat{Z}_1^\Delta - x \right)^2,$$

where again we require that $x = S^*(S(0) + i\Delta)$ for some nonnegative integer i .

Proposition 4.7. *With the assumptions of Theorem 4.5, letting $R < \infty$ be fixed, we have*

$$\lim_{\Delta \downarrow 0} \sup_{|x| < R} \left| b_{\widehat{Z}_\Delta}(x) - b_{Z^*}(x) \right| = 0,$$

where $b_{Z^*}(S^*(S(1))) = 0$ by the absorbing behavior of the boundary at $S^*(S(1))$.

Proof. We have

$$b_{Z^*}(x) = \alpha^2 \frac{\pi^2((S^*)^{-1}(x))}{\Pi^4((S^*)^{-1}(x))} b((S^*)^{-1}(x))$$

for $x \neq S^*(S(1))$. There exists a constant $\delta > 0$ such that for all Δ and all x satisfying $|x| < R$, if we write $x = S^*(S(0) + i\Delta)$ then $i\Delta \geq \delta$. Let

$$A := (S^*(i+1) - S^*(i))^2 \frac{\alpha \cdot C^\Delta}{2\Delta^2 H^\Delta(i)}; \quad B := \frac{(S^*(i+1) - S^*(i))^2}{\Delta^2} \frac{b(i+1)}{2};$$

and

$$C := \frac{(S^*(i-1) - S^*(i))^2}{\Delta^2} \frac{b(i)}{2}; \quad D := -(S^*(i-1) - S^*(i))^2 \frac{\alpha \cdot C^\Delta}{2\Delta^2 H^\Delta(i)}.$$

Note that $b_{\widehat{Z}_\Delta}(x) = A + B + C + D$ for $x \neq S^*(S(1))$.

As in the proof of Proposition 4.6,

$$\lim_{\Delta \downarrow 0} \sup_{|x| < R, x \neq S^*(S(1))} \left| B - \alpha^2 \frac{\pi^2(i)}{2\Pi^4(i)} b(i) \right| = 0;$$

$$\lim_{\Delta \downarrow 0} \sup_{|x| < R, x \neq S^*(S(1))} \left| C - \alpha^2 \frac{\pi^2(i)}{2\Pi^4(i)} b(i) \right| = 0.$$

Rewrite A (with analogous results holding for D) as

$$A = (S^*(i+1) - S^*(i)) \frac{S^*(i+1) - S^*(i)}{\Delta} \frac{\alpha \cdot C^\Delta}{2\Delta H^\Delta(i)}.$$

From the uniform continuity of $(S^*)''(\cdot)$ on bounded intervals, $\frac{S^*(i+1) - S^*(i)}{\Delta}$ converges uniformly for $|x| = |S^*(i)| < R$ to $(S^*)'(i)$, which is uniformly bounded for $|x| = |S^*(i)| < R$. Also, $(C^\Delta)^{-1} 2\Delta H^\Delta(i)$ is bounded away from 0 for $|x| = |S^*(i)| < R$, and $S^*(\cdot)$ is uniformly continuous for $|x| = |S^*(i)| < R$. Hence

$$\lim_{\Delta \downarrow 0} \sup_{|x| < R, x \neq S^*(S(1))} |A| = 0; \quad \lim_{\Delta \downarrow 0} \sup_{|x| < R, x \neq S^*(S(1))} |D| = 0.$$

Lastly, note that $b_{\widehat{Z}_\Delta}(S^*(S(1))) = 0$, which finishes the proof. \square

We are now ready to prove Theorem 4.5:

Proof of Theorem 4.5. With x fixed so that $\mathbb{P}(Z^*)^{-1} = \delta_{S^*(x)}$, let R be such that $|S^*(x)| < R$ and $|S^*(S(1))| < R$, and define

$$\tau_R := \inf\{t \geq 0 : |Z_t^*| = R\}.$$

It follows that $Z^*(\cdot \wedge \tau_R)$ is equal in distribution to the diffusion process with state space $\mathcal{S}_R^* := [-R, S^*(S(1))]$ and generator

$$A_R^* := \frac{1}{2} b_{Z^*}(\cdot) \frac{\partial^2}{\partial x^2}$$

operating on the domain

$$\mathcal{D}_R := \{f \in C(\mathcal{S}_R^*) \cap C^2[(\mathcal{S}_R^*)^\circ] \mid A_R^* f \in C(\mathcal{S}_R^*), A_R^* f(-R) = A_R^* f(S^*(S(1))) = 0\}.$$

Since $b_{Z^*} > 0$ on \mathcal{S}^* , it follows that if $f \in \mathcal{D}_R$ then $f \in C^2(\mathcal{S}_R^*)$. Let

$$\tau_R^\Delta := \inf\{t \geq 0 : |\widehat{Z}_t^\Delta| \geq R\},$$

and define the sequence of absorbing Markov chains $\widehat{V}^\Delta(\cdot) := \widehat{Z}^\Delta(\cdot \wedge \tau_R^\Delta)$ with state space

$$\mathcal{S}_R^{*,\Delta} := \{x \in \mathcal{S}^{*,\Delta} : x \leq \lceil R \rceil_\Delta\}$$

where $\lceil y \rceil_\Delta$ “rounds” y to the smallest element $\geq y$ in the grid $\{S^*(S(0)), S^*(S(0) + \Delta), \dots, S^*(S(1) - \Delta), S^*(S(1))\}$. Lastly define $V^{*,\Delta}(t) := \widehat{V}^\Delta(\lfloor t/h \rfloor)$.

From [8, Corollary 4.8.9], to prove that $V^{*,\Delta}$ converges (as $\Delta \downarrow 0$) to $Z^*(\cdot \wedge \tau_R)$, it suffices to show that for each fixed $f \in \mathcal{D}_R$ we have

$$(4.10) \quad \lim_{\Delta \rightarrow 0} \sup_{x \in \mathcal{S}_R^{*,\Delta}} |\rho_\Delta f(x) - f(x)| = 0 = \lim_{\Delta \rightarrow 0} \sup_{x \in \mathcal{S}_R^{*,\Delta}} |\rho_\Delta A_R^* f(x) - \widehat{A}^\Delta f(x)|$$

where $\widehat{A}^\Delta f(x) := h_\Delta^{-1} [\mathbb{E}_x f(\widehat{V}_1^\Delta) - f(x)]$. The first equality in (4.10) is trivial. Consider the second equality. At $x = -R$ or $x = S^*(S(1))$, we have

$$|\rho_\Delta A_R^* f(x) - \widehat{A}^\Delta f(x)| = 0,$$

since both $\rho_\Delta A_R^* f(x)$ and $\widehat{A}^\Delta f(x)$ equal 0 for $x = -R$ or $x = S^*(S(1))$. For x in the interior of $\mathcal{S}_R^{*,\Delta}$, from a Taylor expansion of f with remainder in intermediate-point form we find

$$(4.11) \quad \left| \widehat{A}^\Delta f(x) - \left[f'(x) a_{\widehat{Z}^\Delta}(x) + \frac{f''(x)}{2} b_{\widehat{Z}^\Delta}(x) \right] \right| \leq \frac{c}{2} b_{\widehat{Z}^\Delta}(x),$$

where, with $x = S^*(S(0) + i\Delta)$, we take

$$c = \max\{|f''(S^*(S(0) + (i-1)\Delta)) - f''(x)|, |f''(S^*(S(0) + (i+1)\Delta) - f''(x))|\}.$$

From (4.11), Proposition 4.6, Proposition 4.7, and the fact that $f \in C^2(\mathcal{S}_R^*)$, we have that $\widehat{A}^\Delta f(x)$ converges uniformly to $A_R^* f(x)$, and so (4.10) is proven.

We have now established that $V^{*,\Delta}$ converges in distribution to $Z^*(\cdot \wedge \tau_R)$. The relative compactness of $V^{*,\Delta}$ follows as in the proof of [8, Theorem 7.4.1], and therefore [30, Theorem 11.1.1] implies $Z^{*,\Delta} \Rightarrow Z^*$. Lastly, noting that $(S^*)^{-1}(\cdot)$ is well-defined and measurable (indeed it is continuous!) we have that

$$\widehat{Y}^\Delta = (S^*)^{-1}(Z^{*,\Delta}) \Rightarrow (S^*)^{-1}(Z^*) = Y^*$$

by [8, Theorem 3.10.2]. □

4.3. Convergence extended to entrance boundary cases. For 0 an entrance boundary of X and 1 reflecting, again consider $Y = S(X)$, a regular diffusion in natural scale on $\mathcal{S} = [-\infty, S(1)]$ begun in $\pi_0 = \pi^{(x)}$, the stationary measure for Y truncated (conditioned) to $(-\infty, x)$ for some $x \in \mathcal{S}$. If $b(\cdot)$ is bounded away from both 0 and ∞ on \mathcal{S} , then the constructions of the approximating primal and dual chains are identical to the case where 0 is reflecting, and details are omitted. However, if $\lim_{x \rightarrow -\infty} b(x) = \infty$, then the approximating sequences of Markov chains need to be defined differently.

To this end, on $\mathcal{S}^\Delta := \{S(1) - i_\Delta \Delta, \dots, S(1) - \Delta, S(1)\}$, with i_Δ chosen so that $i_\Delta \Delta \rightarrow \infty$, define a birth-and-death transition matrix P^Δ via (here using the shorthand i for $S(1) - i\Delta$)

$$P^\Delta(i, i+1) = P^\Delta(i, i-1) := \frac{b(i)h_\Delta}{2\Delta^2} \text{ for } 0 < i < i_\Delta$$

$$P^\Delta(i_\Delta, i_\Delta - 1) := \frac{b(i_\Delta)h_\Delta}{\Delta^2},$$

$$P^\Delta(0, 1) := \frac{b(0)h_\Delta}{\Delta^2},$$

with $P^\Delta(i, i)$ chosen to make the row sums of P^Δ equal to 1, and

$$h_\Delta := \frac{\Delta^2}{4 \cdot \sup_{i \leq i_\Delta} b(i)}$$

chosen again to ensure monotonicity. For an initial probability distribution π_0^Δ on \mathcal{S}^Δ , consider a birth-and-death Markov chain $X^\Delta \sim (\pi_0^\Delta, P^\Delta)$. Let the stationary distribution of X^Δ be denoted π^Δ . Let

$$i_{\Delta, x} := \lfloor [S(1) - x]/\Delta \rfloor,$$

and assume that $\pi_0^\Delta := \pi^{\Delta, i_{\Delta, x}}$ is π^Δ truncated (conditioned) to $\{S(1) - i_\Delta \Delta, \dots, S(1) - i_{\Delta, x} \Delta\}$. Again note that $\pi_0^\Delta \Rightarrow \pi^{(x)}$.

The following theorem is proven in a similar fashion to Theorem 4.1, and so the proof will be sketched with some detail omitted (see Appendix A for notation).

Theorem 4.8. *Assume $b(\cdot)$ is continuous and bounded away from 0 over $(-\infty, S(1)]$. Let $\mathbb{P}Y^*(0)^{-1} = \pi^{(x)}$ and, as above, let $X^\Delta \sim (\pi_0^\Delta, P^\Delta)$ with π_0^Δ equal to π^Δ truncated (conditioned) to $\{S(1) - i_\Delta \Delta, \dots, S(1) - i_{\Delta, x} \Delta\}$. Define the continuous-time stochastic process Y^Δ by setting $Y_t^\Delta := X_{\lfloor t/h_\Delta \rfloor}^\Delta$ for $t \geq 0$. Then $Y^\Delta \Rightarrow Y$.*

Proof. Fix R such that $S(1) < R < \infty$. With

$$\tau_R^\Delta := \inf\{t \geq 0 : |Y_t^\Delta| \geq R\},$$

consider

$$Z^\Delta(\cdot) := Y^\Delta(\cdot \wedge \tau_R^\Delta) = X^\Delta\left(\left\lfloor \frac{\cdot \wedge \tau_R^\Delta}{h_\Delta} \right\rfloor\right).$$

With

$$\tau_R := \inf\{t \geq 0 : |Y_t| = R\},$$

let $Z(\cdot) := Y(\cdot \wedge \tau_R)$. Denote the generator of Z by A_Z , with domain $\mathcal{D}(A_Z)$. Writing (T_R^Δ) for the transition semigroup associated with the Markov chain X^Δ

absorbed at absolute value R , let $A_R^\Delta := h_\Delta^{-1}(T_R^\Delta - I)$. Just as we showed (B.1) in the proof of Theorem 4.1, here we can show that

$$(4.12) \quad \lim_{\Delta \rightarrow \infty} \sup_{x \in [-R, S(1)]} |(A_R^\Delta \rho_\Delta f)(x) - (A_Z f)(x)| = 0$$

for all $f \in \mathcal{D}(A_Z)$.

By [8, Corollary 4.8.9], we have (see Appendix A) that $Z^\Delta \Rightarrow Z$. The proof is finished by applying [30, Theorem 11.1.1] to see that $Y^\Delta \Rightarrow Y$ as desired. \square

Let Y^* be a SSD of Y , and let Z^* be Y^* put into natural scale. Form the dual Markov chain to X^Δ , and denote the dual by $\hat{X}^\Delta \sim (\delta_x, \hat{P}^\Delta)$. The following proposition gives the dual-convergence theorem analogous to Theorem 4.5.

Theorem 4.9. *With the same assumptions as in Theorem 4.8, further assume $b(\cdot) \in C^2(-\infty, S(1)]$ and*

$$(4.13) \quad \inf_{y \in \mathcal{S}} y^4 m(M^{-1}(-1/y)) > 0.$$

For $t \geq 0$, define $\hat{Y}^\Delta(t) := \hat{X}^\Delta(\lfloor t/h_\Delta \rfloor)$. Then $\hat{Y}^\Delta \Rightarrow Y^*$.

Proof. The proof follows along the same path as the proof of Theorem 4.5 and so details are omitted. The only wrinkle here is the assumption (4.13), which is a technical condition needed to make the infinitesimal variance of the dual diffusion in natural scale bounded away from 0, which we exploited in the proof of Theorem 4.5. \square

Remark 4.10. Under some mild assumptions, the above theory can easily be extended to the case where both 0 and 1 are entrance boundaries for X . For example, if X is in natural scale, it is sufficient that b_X is bounded away from 0 and twice continuously differentiable on \mathbb{R} . The analogues of Theorem 4.1 and Theorem 4.5 can be easily recovered. Details are omitted.

5. SEPARATION AND HITTING TIMES

In the Markov chain setting, strong stationary duality gives that the separation mixing time in the primal chain is equal in law to a suitable absorption time in the dual chain. By studying and bounding the absorption time, which is sometimes more tractable than direct consideration of the mixing time, we can tightly bound the separation mixing time in our primal chain. See [4] for further detail. Spelling this out more fully, if $X \sim (\pi_0, P)$ is an ergodic discrete-time Markov chain with state space S , stationary distribution π , and with SSD (as defined in [4]) $X^* \sim (\pi_0^*, P^*)$ absorbing in m , then for every t we have

$$(5.1) \quad \text{sep}(t) := \sup_{i \in S} \left(1 - \frac{\pi_t(i)}{\pi(i)} \right) \leq \mathbb{P}_{\pi_0^*}(T_m^* > t).$$

Under some monotonicity conditions, for example if the primal is a MLR chain on a linearly ordered state space, the inequality in (5.1) can be made to be an equality for every t by a suitable formation of the dual chain.

In our present diffusion setting, with X a regular diffusion on $[0, 1]$ with either reflecting or entrance behavior at the boundaries, we would like to recover a result similar to (5.1). Let Π be the invariant distribution for X , let $X_0 \sim \Pi_0$, and, given $t > 0$, let Π_t be the corresponding distribution of X_t . If $\Pi_t \ll \Pi$, define

$$a(t) := \text{ess inf } R_t = \sup \{r \mid \Pi(R_t < r) = 0\}$$

to be the essential infimum (with respect to Π) of (any version of) the Radon–Nikodym derivative $R_t := d\Pi_t/d\Pi$. We define the *separation* of the diffusion from Π at time t as follows:

$$(5.2) \quad \text{sep}(\pi_t, \pi) := 1 - a(t).$$

To simplify the notation, we shall write $\text{sep}(t)$ for $\text{sep}(\pi_t, \pi)$ unless the full notation is needed to avoid confusion.

Claim 5.1. *Let $\text{sep}(t) = \text{sep}(\pi_t, \pi)$ be defined as above. Then*

- (a) *We have $0 \leq \text{sep}(t) \leq 1$.*
- (b) *For each t we have $\text{sep}(t) = 0$ if and only if $\Pi_t = \Pi$.*
- (c) *For any Π_0 we have $\Pi_t \ll \Pi$ for all $t > 0$.*
- (d) *The separation $\text{sep}(t)$ is non-increasing in t .*

Proof. For (a), we show equivalently that $0 \leq a(t) \leq 1$. To this end, let R_t be (any version of) the Radon–Nikodym derivative $d\Pi_t/d\Pi$. Since $R_t(y) \geq 0$ for all y , we have $a(t) \geq 0$. But also

$$(5.3) \quad 1 = \int_0^1 \Pi_t(dy) = \int_0^1 R_t(y) \Pi(dy) \geq a(t) \int_0^1 \Pi(dy) = a(t),$$

finishing the proof.

For (b), note that if $\Pi_t = \Pi$, we can take $R_t \equiv 1$ as a version of the Radon–Nikodym derivative $d\Pi_t/d\Pi$, and thence $\text{sep}(t) = 0$. Conversely, if $\text{sep}(t) = 0$, then $a(t) = 1$ and (5.3) is an equality; therefore $R_t = 1$ almost surely with respect to Π , and so $\Pi_t = \Pi$.

For (c), let $x \in (0, 1)$. When $\Pi_0 = \delta_x$, regularity of X guarantees the existence of a density for Π_t with respect to Π , call it $f_x(\cdot)$. For any Π_0 , it follows that the Π_0 -mixture of the densities $f_x(\cdot)$ is a density for Π_t with respect to Π [and so $\text{sep}(t)$ is well defined].

For (d), for each $s > 0$ let $R_s = d\Pi_s/d\Pi$. Let $0 < t < u$ and note for any $A \in \mathcal{B}$, the Borel σ -field of $[0, 1]$, that

$$\begin{aligned} \int_A R_u(y) \Pi(dy) &= \Pi_u(A) = \int_0^1 P_{u-t}(x, A) \Pi_t(dx) = \int_0^1 R_t(x) P_{u-t}(x, A) \Pi(dx) \\ &\geq a(t) \int_0^1 P_{u-t}(x, A) \Pi(dx) = a(t) \Pi(A) = \int_A a(t) \Pi(dy). \end{aligned}$$

Hence $R_u \geq a(t)$ almost surely with respect to Π . Hence $a(u) \geq a(t)$, and therefore $\text{sep}(u) \leq \text{sep}(t)$, as desired. \square

As in the discrete setting, we are able to bound $\text{sep}(t)$ in our primal diffusion X using the absorption time in state 1 of our dual diffusion. In the diffusion setting, by virtue of diffusions being stochastically monotone, the inequality in (5.1) is an equality without needing further assumptions. Spelling this out:

Lemma 5.2. *Let X be a regular diffusion on $[0, 1]$ begun in Π_0 , let X have either reflecting or entrance behavior at the boundaries, and let Π be the stationary measure for X . Let T_1^* be the hitting time of state 1 in the SSD diffusion X_t^* (as defined in Definition 3.1) begun in Π_0^* satisfying (3.3). Then*

$$\text{sep}(t) = \mathbb{P}_{\Pi_0^*}(T_1^* > t) = 1 - \mathbb{P}_{\Pi_0^*}(X_t^* = 1).$$

Proof. Let $f \in F[0, 1]$. By Remark 3.12, we have for all $t > 0$ that $(\Pi_t, f) = (\Pi_t^*, \Lambda f)$. Therefore, writing $R_t = d\Pi_t/d\Pi$ as usual, we have

$$\begin{aligned} \int_{[0,1]} \pi(x) R_t(x) f(x) dx &= \int_{[0,1]} \Pi(dx) R_t(x) f(x) \\ &= \int_{[0,1]} \Pi_t(dx) f(x) \\ &= \int_{[0,1]} \Pi_t^*(dx) \int_{[0,x]} \pi^{(x)}(y) f(y) dy \\ &= \int_{[0,1]} \int_{[y,1]} \Pi_t^*(dx) \pi^{(x)}(y) f(y) dy. \end{aligned}$$

This holds for all $f \in F[0, 1]$, and so

$$(5.4) \quad R_t(y) = \int_{[y,1]} \frac{\Pi_t^*(dx)}{\Pi(x)}$$

for Lebesgue-a.e. (i.e., for Π -a.e.) y . Thus $\Pi(R_t < r) = 0$ if and only if the right side of (5.4) is at least r for Π -a.e. y , or, equivalently, $\Pi_t^*(\{1\})/\Pi(1) = \Pi_t^*(\{1\}) \geq r$. Therefore $a(t) = \Pi_t^*(\{1\}) = \mathbb{P}_{\Pi_0^*}(X_t^* = 1)$ and so $\text{sep}(t) = 1 - \mathbb{P}_{\Pi_0^*}(X_t^* = 1)$. \square

Remark 5.3. We can also prove Lemma 5.2 by passing to the limit the corresponding discrete-time results for the Markov chains in Section 4. First, suppose that $Y_0 \sim \Pi^{(x)}$ for some $x > 0$ and hence $\Pi_0^* = \delta_x$ (see Remark 3.3). Adopting the notation of Section 4, the primal birth-and-death Markov chain $X^\Delta \sim (\pi_0^\Delta, P^\Delta)$ has

$$\text{sep}^\Delta(t) = \sup_i \left(1 - \frac{\sum_j \pi_0^\Delta(j) P_t^\Delta(j, i)}{\pi^\Delta(i)} \right).$$

Now

$$\begin{aligned} \frac{\sum_j \pi_0^\Delta(j) P_t^\Delta(j, i)}{\pi^\Delta(i)} &= \frac{1}{H^\Delta(i_{\Delta, x})} \sum_{j \leq i_{\Delta, x}} \frac{\pi^\Delta(j) P_t^\Delta(j, i)}{\pi^\Delta(i)} \\ &= \frac{1}{H^\Delta(i_{\Delta, x})} \mathbb{P}_i(X_t^\Delta \leq i_{\Delta, x}). \end{aligned}$$

The monotonicity conditions outlined in Remark 4.4 and [4, Remark 4.15] imply that this last expression is minimized (for each $t = 0, 1, \dots$) when $i = n^\Delta$, and that the minimum value is

$$(5.5) \quad \frac{1}{H^\Delta(i_{\Delta, x})} \mathbb{P}_{n^\Delta}(X_t^\Delta \leq i_{\Delta, x}) = 1 - \text{sep}^\Delta(t) = \mathbb{P}_{i_{\Delta, x}}(\widehat{T}_{n^\Delta} \leq t),$$

where \widehat{X}^Δ is the strong stationary dual of X^Δ as defined at (4.4)–(4.7), with absorption time \widehat{T}_{n^Δ} in its largest state n^Δ . We now substitute $\lfloor t/h \rfloor$ for t , and recall that $h \equiv h_\Delta$ is a function of Δ and that $Y_t^\Delta := X_{\lfloor t/h \rfloor}^\Delta$ (and analogously for \widehat{Y}_t^Δ), to find for real $t \geq 0$ that

$$(5.6) \quad \frac{1}{H^\Delta(i_{\Delta, x})} \mathbb{P}_{S(1)}(Y_t^\Delta \leq S(0) + i_{\Delta, x} \Delta) = 1 - \text{sep}^\Delta(t) = \mathbb{P}_{i_{\Delta, x}}(\widehat{Y}_t^\Delta = S(1)),$$

where $i_{\Delta, x}$ is short for $S(0) + i_{\Delta, x} \Delta$.

By Theorem 4.1, the left side of (5.6) converges to $\frac{1}{\Pi(x)}\mathbb{P}_{S(1)}(Y_t \leq x)$ [where we note that the hypothesis of Theorem 4.1 is met for the deterministic initial conditions $Y_0^\Delta = Y_0 = S(1)$]. Theorem 4.5 implies that

$$(5.7) \quad \lim_{\Delta \downarrow 0} \mathbb{P}_{i_{\Delta,x}}(\hat{Y}_t^\Delta > S(1) - \varepsilon) = \mathbb{P}_x(Y_t^* > S(1) - \varepsilon).$$

Let \check{X}^Δ be the Siegmund dual of (the time-reversal of) X^Δ ; by definition, \check{X}^Δ is a Markov chain satisfying

$$\mathbb{P}_y(X_t^\Delta \leq z) = \mathbb{P}_z(y \leq \check{X}_t^\Delta)$$

for all $y, z \in \mathcal{S}^\Delta$ and $t = 0, 1, 2, \dots$. Equation (5.3) in [4] gives, with $h = h_\Delta$ and with $\lceil x \rceil_\Delta$ (respectively, $\lfloor x \rfloor_\Delta$) being the smallest element $\geq x$ (resp., the largest element $\leq x$) in the grid $\{S^*(S(0)), S^*(S(0) + \Delta), \dots, S^*(S(1) - \Delta), S^*(S(1))\}$, that

$$\begin{aligned} \mathbb{P}_{i_{\Delta,x}}(\hat{Y}_t^\Delta > S(1) - \varepsilon) &= \mathbb{P}_{i_{\Delta,x}}(\hat{X}_{\lfloor t/h \rfloor}^\Delta > S(1) - \varepsilon) \\ &= \sum_{j > S(1) - \varepsilon} \frac{H^\Delta(j)}{H^\Delta(i_{\Delta,x})} \mathbb{P}_{i_{\Delta,x}}(\check{X}_{\lfloor t/h \rfloor}^\Delta = j) \\ &\geq \frac{H^\Delta(\lfloor S(1) - \varepsilon \rfloor_\Delta)}{H^\Delta(i_{\Delta,x})} \mathbb{P}_{i_{\Delta,x}}(\check{X}_{\lfloor t/h \rfloor}^\Delta > \lceil S(1) - \varepsilon \rceil_\Delta) \\ &\geq \frac{H^\Delta(\lfloor S(1) - \varepsilon \rfloor_\Delta)}{H^\Delta(i_{\Delta,x})} \mathbb{P}_{i_{\Delta,x}}(\check{X}_{\lfloor t/h \rfloor}^\Delta \geq \lceil S(1) - 2\varepsilon \rceil_\Delta) \\ &= \frac{H^\Delta(\lfloor S(1) - \varepsilon \rfloor_\Delta)}{H^\Delta(i_{\Delta,x})} \mathbb{P}_{\lceil S(1) - 2\varepsilon \rceil_\Delta}(X_{\lfloor t/h \rfloor}^\Delta \leq i_{\Delta,x}) \\ &\rightarrow \frac{\Pi(S(1) - \varepsilon)}{\Pi(x)} \mathbb{P}_{S(1) - 2\varepsilon}(Y_t \leq x) \text{ as } \Delta \downarrow 0, \end{aligned}$$

and this last expression converges to $\frac{1}{\Pi(x)}\mathbb{P}_{S(1)}(Y_t \leq x)$ as $\varepsilon \downarrow 0$. We can also get an upper bound on $\mathbb{P}_{i_{\Delta,x}}(\hat{Y}_t^\Delta > S(1) - \varepsilon)$ using

$$\begin{aligned} \sum_{j > S(1) - \varepsilon} \frac{H^\Delta(j)}{H^\Delta(i_{\Delta,x})} \mathbb{P}_{i_{\Delta,x}}(\check{X}_{\lfloor t/h \rfloor}^\Delta = j) &\leq \frac{1}{H^\Delta(i_{\Delta,x})} \mathbb{P}_{i_{\Delta,x}}(\check{X}_{\lfloor t/h \rfloor}^\Delta \geq \lfloor S(1) - \varepsilon \rfloor_\Delta) \\ &= \frac{1}{H^\Delta(i_{\Delta,x})} \mathbb{P}_{\lfloor S(1) - \varepsilon \rfloor_\Delta}(X_{\lfloor t/h \rfloor}^\Delta \leq i_{\Delta,x}) \\ &\rightarrow \frac{1}{\Pi(x)} \mathbb{P}_{\lfloor S(1) - \varepsilon \rfloor_\Delta}(Y_t \leq x) \text{ as } \Delta \downarrow 0, \end{aligned}$$

and this last expression converges to $\frac{1}{\Pi(x)}\mathbb{P}_{S(1)}(Y_t \leq x)$ as $\varepsilon \downarrow 0$. Letting $\varepsilon \downarrow 0$ in (5.7) gives

$$\frac{1}{\Pi(x)} \mathbb{P}_{S(1)}(Y_t \leq x) = \mathbb{P}_x(T_{S(1)}^* \leq t).$$

Now $\Pi_t \ll \Pi$ for all $t > 0$; let $R_t = d\Pi_t/d\Pi$, so that for any $A = [S(0), s)$ with $s \in \mathcal{S}$ we have

$$\mathbb{P}_{\Pi(x)}(Y_t \in A) = \int_A R_t(y) \Pi(dy),$$

and also

$$\begin{aligned}\mathbb{P}_{\Pi(x)}(Y_t \in A) &= \int_{[S(0), x]} \frac{\Pi(dy)}{\Pi(x)} P_t(y, A) \\ &= \int_{[S(0), x]} \frac{1}{\Pi(x)} \pi(y) \int_A p_t(y, z) dz dy.\end{aligned}$$

We will now appeal to the reversibility of Y . A diffusion process X with generator A and state space I is *reversible* with respect to the distribution μ if for all $f, g \in \mathcal{D}_A$ we have

$$(5.8) \quad \int f(y) (Ag)(y) \mu(dy) = \int (Af)(y) g(y) \mu(dy).$$

If Y satisfies the assumptions of Lemma 5.2, noting that $f, g \in \mathcal{D}_A$ implies that the derivatives of each function vanish at the boundary of the state space, integration by parts yields that (5.8) holds for $\mu = \Pi$, the stationary distribution of Y , and the primal diffusion is reversible with respect to Π . Also note that (5.8) is equivalent to the following (see [22, Section II.5]): for all $f, g \in C(S)$, and for all $t > 0$ we have

$$(5.9) \quad \int f(y) (T_t g)(y) \Pi(dy) = \int (T_t f)(y) g(y) \Pi(dy),$$

where (T_t) is the one parameter semigroup associated with Y .

Letting f and g be suitably continuous approximations of $\mathbb{1}([S(0), x])$ and $\mathbb{1}(A)$, and appealing to (5.9), we have

$$\begin{aligned}\int_{[S(0), x]} \frac{1}{\Pi(x)} \pi(y) \int_A p_t(y, z) dz dy &= \int_{[S(0), x]} \frac{1}{\Pi(x)} \int_A p_t(z, y) \pi(z) dz dy \\ &= \int_A \frac{1}{\Pi(x)} \mathbb{P}_z(Y_t \leq x) \Pi(dz),\end{aligned}$$

and so $\frac{1}{\Pi(x)} \mathbb{P}_z(Y_t \leq x)$ is a version of $R_t(z)$. By monotonicity of Y , we have that $\frac{1}{\Pi(x)} \mathbb{P}_z(Y_t \leq x)$ is minimized when $z = S(1)$, and hence for Y we have

$$a(t) = 1 - \text{sep}(t) = \frac{1}{\Pi(x)} \mathbb{P}_{S(1)}(Y_t \leq x) = \mathbb{P}_x(T_{S(1)}^* \leq t),$$

establishing Lemma 5.2 for Y .

Remark 5.4. In the Markov chain setting of [4] and [9], the authors were able to justify their “strong stationary duality” nomenclature by tying their then-new notion of duality to the more classical notions of duality in the stochastic process literature. Specifically, let $X \sim (\pi_0, P)$ be an ergodic Markov chain with stationary distribution π . If X satisfies specific monotonicity conditions, namely, that the time reversal \tilde{P} is monotone and $\pi_0(x)/\pi(x)$ decreases in x , then with H be cumulative of π , they show that the SSD X^* of X is the Doob H -transform of the Siegmund dual of the time-reversal of X .

For a Markov process Y with transition operator $P_t(x, dy)$, the *Doob H -transform* of Y is the right-continuous Markov process with transition operator

$$Q_t(x, dy) := \frac{H(y)}{H(x)} P_t(x, dy).$$

It has played a central role in Markov process duality theory, especially in the context of processes conditioned to die in a given set or point. See [28, Chapter

VII] for further detail. The *Siegmund dual* of a Markov process Y with state space \mathcal{S} is a Markov process Z on \mathcal{S} satisfying:

$$\mathbb{P}_y(Y_t \leq z) = \mathbb{P}_z(y \leq Z_t) \text{ for all } y, z \in \mathcal{S}.$$

It has played a prominent role in the study of birth-and-death chains and diffusion theory and in the study of interacting particle systems (see [22, Section II.3] for extensive background).

To justify the nomenclature in the present diffusion setting, consider the diffusion X as defined in Section 2, and let X^* be the strong stationary dual of X specified in Definition 3.1. Then, recalling from Remark 3.6 that for all $f \in F[0, 1]$ we have $\Lambda T_t f = T_t^* \Lambda f$, a simple calculation yields

$$\int_{[0, x]} p_t(z, y) dy = \int_{[z, 1]} \frac{\Pi(x)}{\Pi(y)} P_x^*(X_t^* \in dy),$$

giving us immediately that X^* is the Doob H -transform of the Siegmund dual of (the time reversal of) X , where H here is the cumulative stationary distribution Π .

A functional definition of duality generalizing Siegmund's definition was introduced in [15]. For extensive background see again [22, Section II.3]. Briefly, let X and Y be two Markov processes with state spaces \mathcal{S} and \mathcal{S}' and let f be a bounded measurable function on $\mathcal{S} \times \mathcal{S}'$. We define Y to be the *dual* of X with respect to the function f if

$$\mathbb{E}_x f(X_t, y) = \mathbb{E}_y f(x, Y_t), \quad \text{for all } x \in \mathcal{S}, y \in \mathcal{S}'.$$

As in [4, Theorem 5.12], a simple calculation yields that, in the diffusion setting, X and its SSD X^* are dual with respect to the function

$$f(x, x^*) := \begin{cases} 1/\Pi(x^*), & \text{if } x \leq x^* \\ 0, & \text{otherwise,} \end{cases}$$

on $I \times I$, further justifying the duality name for X^* .

With [4, Definition 5.16], the authors generalized the classical notion of functional duality. Adapted to the present setting, let X and Y be two diffusions defined on a common probability space with state spaces \mathcal{S} and \mathcal{S}' . We say Y is *dual* to X with respect to a function $f : \mathcal{S} \times \mathcal{S}' \rightarrow \mathbb{R}$ and distribution μ on $\mathcal{S} \times \mathcal{S}'$ if

$$\mathbb{E}_\mu f(X_t, Y_0) = \mathbb{E}_\mu f(X_0, Y_t).$$

In [4, Theorem 5.19], the authors were able to show that the strong stationary dual of an ergodic Markov chain $X \sim (\pi_0, P)$ with stationary distribution π , and with the additional properties that the time reversal \tilde{P} is monotone and $\pi_0(x)/\pi(x)$ decreases in x , is dual to the primal chain with respect to this new functional definition, for suitable choices of f and μ . We are able to recover the analogue of their Theorem 5.19 here, as it is easy to see that X^* (the strong stationary dual of X) and X are dual with respect to the function $f(x^*, x) = \mathbf{1}(x \leq x^*)\pi(x)/\Pi(x^*)$ and μ equal to any mixture of the distributions $\delta_{x^*} \times \Pi(x^*)$ with $x^* \in [0, 1]$.

6. HITTING TIMES AND EIGENVALUES

In the continuous-time birth-and-death chain setting, a famous theorem due to Karlin and MacGregor [17] asserts that the hitting time of state n for a birth-and-death chain X on $\{0, 1, \dots, n\}$ started in state 0 is distributed as the sum of

independent exponential random variables with parameters relating to the eigenvalues of the generator of X . Fill [9] used strong stationary duality to exploit Karlin and MacGregor's result to prove that the separation from stationarity for an ergodic continuous-time birth-and-death chain X at time t is equal to $\mathbb{P}(Y > t)$ where Y is a sum of independent exponential random variables with parameters depending on the eigenvalues of the generator of X . In [6], Diaconis and Saloff-Coste used Fill's result and tight concentration bounds on the tail probabilities of Y to prove the existence of a separation cutoff for a sequence (X_n) of birth-and-death chains under certain conditions on the eigenvalues of the generators of the chains X_n . In this section, we outline and recover the analogous theory in the diffusion setting.

To this effect, consider again a diffusion X on $[0, 1]$ with generator A , and with reflecting or entrance boundary behavior at each boundary, satisfying the assumptions of Theorem 3.4. Let X^* be a strong stationary dual of X according to Definition 3.1. For fixed λ , let $v_\lambda(x)$ be the solution to the eigenvalue problem associated with A (respectively, A^*):

$$(6.1) \quad Av + \lambda v = 0 \quad (A^*v + \lambda v = 0)$$

with boundary condition

$$(6.2) \quad B_0(v) = 0$$

where B_0 represents the following boundary condition:

$$B_0(v) := \begin{cases} v(0), & \text{if } 0 \text{ is absorbing or exit;} \\ \frac{dv}{ds}^+(0), & \text{if } 0 \text{ is instantaneously reflecting or entrance.} \end{cases}$$

Let $T_{x,y}$ be the hitting time of y for X begun in x . From [16, Section 4.6], we have that $v_\lambda(x)$ is unique up to multiplicative constant and that the moment generating function of $T_{x,y}$, call it $\psi_{x,y}$, can be expressed as

$$(6.3) \quad \psi_{x,y}(\lambda) = v_\lambda(x)/v_\lambda(y).$$

A completely analogous set of results hold for A^* .

If we further add the relevant boundary condition at 1, namely that $B_1(v) = 0$ (where B_1 is defined analogously to B_0), then we have from Sturm–Liouville theory (see for example [19, Theorem 4.1]) that the eigenvalues of A^* (resp., nonzero eigenvalues of A) satisfying (6.1) with the two boundary conditions are countable, real, positive, and simple and can be ordered such that

$$0 < \lambda_1 < \lambda_2 < \cdots \uparrow \infty;$$

further, they satisfy $\sum_{k=1}^{\infty} \lambda_k^{-1} < \infty$. For extensive background on the relevant Sturm–Liouville theory, see for example [31]. The eigenfunctions and eigenvalues of A and A^* are connected by the following simple relationship:

Proposition 6.1. *Adopt the same assumptions as Theorem 3.4, and further assume that $b(\cdot) > 0$ on $(0, 1)$ and that 1 is a reflecting boundary for X and 0 is either a reflecting or entrance boundary for X . Fix $\lambda > 0$.*

(a) *Suppose that $v = f$ is a solution of (6.1) for generator A with boundary conditions $B_0(v) = B_1(v) = 0$. Then $v = \Lambda f$ is a solution of (6.1) for generator A^* with boundary conditions $B_0^*(v) = B_1^*(v) = 0$ (and the same λ).*

(b) Suppose that $v = g$ is a solution of (6.1) for generator A^* with boundary conditions $B_0^*(v) = B_1^*(v) = 0$. Then $f(\cdot) = g(\cdot) + \frac{\Pi(\cdot)}{\pi(\cdot)}g'(\cdot)$ is a solution of (6.1) for generator A with boundary conditions $B_0(v) = B_1(v) = 0$ (and the same λ).

Proof. (a) If $f(\cdot)$ satisfies (6.1) for A and the boundary conditions $B_0(f) = B_1(f) = 0$, then $f \in \mathcal{D}_A$ and

$$\frac{df}{dS}^+(0) = \left(\frac{f'}{s}\right)^+(0) = 0 = \frac{df}{dS}^-(1) = \left(\frac{f'}{s}\right)^-(1).$$

From (3.1) we have $\Lambda f \in \mathcal{D}_{A^*}$, and from (3.2) we have

$$(6.4) \quad A^* \Lambda f(\cdot) = \Lambda A f(\cdot) = \Lambda(-\lambda f)(\cdot) = -\lambda \Lambda f(\cdot)$$

on I . Therefore, $\Lambda f(\cdot)$ satisfies (6.1) for A^* . Also $B_0^*(\Lambda f) = 0$ as 0 is an entrance boundary for the dual and $\Lambda f \in \mathcal{D}_{A^*}$, and similarly $B_1^*(\Lambda f) = 0$.

(b) Note that if $g(\cdot)$ satisfies (6.1) for A^* , then $g \in \mathcal{D}_{A^*}$, and hence $g \in C[0, 1]$, and $g(1) = 0$. Next, on $(0, 1)$ note

$$\begin{aligned} f' &= g' + \frac{\pi^2 - \Pi\pi'}{\pi^2}g' + \frac{\Pi}{\pi}g'' = 2g' - \frac{\Pi\pi'}{\pi^2}g' + \frac{\Pi}{\pi}g'' \\ &= 2g' + \frac{\Pi}{\pi} \left(g'' + \frac{b'}{b}g' - \frac{2a}{b}g' \right) = 2g' + \frac{\Pi}{\pi} \left(-\frac{2\lambda g}{b} - \frac{2\pi}{\Pi}g' \right) \\ &= -2\lambda g \frac{\Pi}{\pi} \frac{1}{b} = -2\lambda g M s \end{aligned}$$

where the fourth equality follows from (6.1). We have that $M^+(0) = 0 = g(1)$ and $M^-(1) < \infty$, and hence

$$\left(\frac{f'}{s}\right)^+(0) = 0 = \left(\frac{f'}{s}\right)^-(1),$$

and therefore $B_0(f) = B_1(f) = 0$. Next, on $(0, 1)$ note

$$\begin{aligned} f'' &= \frac{-2\lambda}{b} \left(\frac{\Pi}{\pi}g' + g \frac{\pi^2 - \Pi\pi'}{\pi^2} - g \frac{\Pi b'}{\pi b} \right) \\ &= \frac{-2\lambda}{b} \left(\frac{\Pi}{\pi}g' + g + \frac{\Pi}{\pi}g \frac{s'}{s} \right) \in C(0, 1), \end{aligned}$$

and hence $f \in C^2(0, 1)$. Combining the above, on $(0, 1)$ we have

$$\begin{aligned} a f' + \frac{1}{2} b f'' &= \lambda g \frac{\Pi}{\pi} \frac{s'}{s} - \lambda \left(\frac{\Pi}{\pi}g' + g + \frac{\Pi}{\pi}g \frac{s'}{s} \right) \\ &= -\lambda \left(\frac{\Pi}{\pi}g' + g \right) = -\lambda f. \end{aligned}$$

To show that $f \in \mathcal{D}_A$ and that f satisfies (6.1) for A with the relevant boundary conditions it remains only to show that $f \in C[0, 1]$. We have (by Theorem 3.4) that 0 is an entrance boundary for X^* , and hence for any fixed $\xi \in (0, 1)$ we have that $N(0) < \infty$ and hence

$$-\int_{(0, \xi]} S^*(\eta) M^*(d\eta) = \int_{(0, \xi]} \frac{1}{M(\eta)} s(\eta) M^2(\eta) d\eta = \int_{(0, \xi]} s(\eta) M(\eta) d\eta < \infty,$$

where the first equality holds by (3.8)–(3.9). Since $g \in C[0, 1]$, it follows that

$$\int_{(0, \xi]} |-2\lambda g(\eta)s(\eta)M(\eta)| d\eta = \int_{(0, \xi]} |f'(\eta)| d\eta < \infty.$$

Hence, by the dominated convergence theorem,

$$\int_{(\omega, \xi]} f'(\eta) d\eta = f(\xi) - f(\omega)$$

has a finite limit as $\omega \downarrow 0$. We conclude that $f \in C[0, 1]$. We have by assumption that 1 is a reflecting boundary for X and hence for any fixed $\xi \in (0, 1)$ we have that $\Sigma(1) < \infty$ and hence

$$\int_{[\xi, 1]} s(\eta)M(\eta) d\eta < \infty.$$

By the same argument that showed that f is continuous at 0, we find that f is also continuous at 1. The proof is finished, as we have established that $f \in C[0, 1]$. \square

In the diffusions setting, we have an analogue (namely [19, Theorem 5.1]) of Karlin and MacGregor's famous result on the eigenvalue expansion on birth-and-death hitting times. Adapted to the present setting, we state the analogue as follows:

Theorem 6.2. *Let X be a regular diffusion process on $[0, 1]$ and assume 0 is either instantaneously reflecting or entrance. Then*

$$(6.5) \quad \lim_{x \rightarrow 0} \psi_{x,1}(\lambda) = \prod_{k=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_k}\right)^{-1},$$

which is the moment generating function of an infinite sum of independent exponential random variables with parameters λ_k .

Combining this with Proposition 6.1 and Lemma 5.2, we arrive at

Theorem 6.3. *Let X be a diffusion on $[0, 1]$ with $X_0 = 0$, with generator A , and with either reflecting or entrance behavior at the boundary 0 and reflecting behavior at the boundary 1. Let the eigenpairs $(\lambda_i, v_{\lambda_i})$, $i = 1, 2, \dots$, of A with $\lambda_i > 0$ satisfying (6.1) and boundary conditions $B_0(v_{\lambda_i}) = 0 = B_1(v_{\lambda_i})$ be labeled so that $0 < \lambda_1 < \lambda_2 < \dots$. Let X^* be a strong stationary dual of X with generator A^* , and note that $X_0^* = 0$ by Remark 3.3. Let W_1, W_2, \dots be independent random variables with $W_i \sim \text{Exp}(\lambda_i)$. Then*

$$\text{sep}(t) = \mathbb{P}_0(T_1^* > t) = \mathbb{P}(W > t) \text{ where } W \stackrel{\mathcal{L}}{=} \sum_{i=1}^{\infty} W_i.$$

This mirrors the corresponding result for birth-and-death Markov chains given by [4, Theorem 4.20] in discrete time and by [9, Theorem 5] in continuous time.

In [6], the authors used [4, Theorem 4.20] to determine conditions for a separation cut-off to occur in a sequence of birth-and-death Markov chains. We shall presently derive analogous results for diffusions using Theorem 6.3. Consider now a sequence of diffusion generators $(A_n)_{n=1}^{\infty}$ defining a sequence of diffusions $(X^n)_{n=1}^{\infty}$ with $X_0^n \sim \nu^n$, on finite intervals $[l_1, r_1] = I_1$, $[l_2, r_2] = I_2$, \dots where all left boundary points, l_n are assumed to be reflecting or entrance and all right boundary points r_n are assumed to be reflecting. Note that without loss of generality we can take

$I_n = [0, r_n]$ for all $n \geq 1$. We write π^n for the stationary distribution for X^n , and we write ν_t^n for the distribution of X^n at time t . This sequence of diffusions exhibits a *separation cut-off at (t_n)* if the sequence (t_n) is such that for any $\epsilon \in (0, 1)$ we have

- (i) $\lim_{n \rightarrow \infty} \text{sep}(\nu_{(1+\epsilon)t_n}^n, \pi^n) = 0$, and
- (ii) $\lim_{n \rightarrow \infty} \text{sep}(\nu_{(1-\epsilon)t_n}^n, \pi^n) = 1$.

To apply Theorem 6.3 here, let the nonzero eigenvalues of A_n be labeled $0 < \lambda_{n,1} < \lambda_{n,2} < \dots$, and let $\nu^n = \delta_0$ for all $n \geq 1$. We further assume that each A_n satisfies the assumptions of Theorem 3.4, and let $(A_n^*)_{n=1}^\infty$ be the sequence of generators of the strong stationary duals of $(X^n)_{n=1}^\infty$ as defined by Definition 3.1. For each $n \geq 1$, let $W_{n,j} \sim \text{Exp}(\lambda_{n,j})$ be independent random variables, and let $W_n \stackrel{\mathcal{L}}{=} \sum_{j=1}^\infty W_{n,j}$. From Theorem 6.3, we have $\text{sep}^n(t) = \mathbb{P}(W_n > t)$. We can therefore get sharp bounds on separation by deriving sharp bounds for the tail probabilities of W_n . To this end, note that we have

$$\mathbb{E} W_n = \sum_{j=1}^\infty \lambda_{n,j}^{-1} < \infty, \quad \text{Var } W_n = \sum_{j=1}^\infty \lambda_{n,j}^{-2} < \infty.$$

An application of the one-sided Chebyshev's inequality gives the analogue to the separation cut-off result [6, Theorem 5.1]:

Theorem 6.4. *Let $(A_n)_{n=1}^\infty$ be a sequence of diffusion generators defining diffusions $(X^n)_{n=1}^\infty$, with $X_0^n \sim \nu^n$, on finite intervals $[0, r_1] = I_1$, $[0, r_2] = I_2$, \dots , where 0 is assumed to be reflecting or entrance for all n , and all right boundary points r_n are assumed to be reflecting. With the eigenvalues $\lambda_{n,i}$ defined as above, this sequence of diffusions exhibits a separation cut-off if and only if*

$$\lim_{n \rightarrow \infty} \lambda_{n,1} \mathbb{E} W_n = \infty,$$

in which case there is a separation cut-off at (t_n) with $t_n := \mathbb{E} W_n$. Further, for any $c > 0$ the following separation bounds hold for any sequence (t_n) , where we restrict to $c \leq 1$ in the second bound:

$$\text{sep}(\nu_{(1+c)t_n}^n, \pi^n) \leq \frac{1}{1 + c^2 \lambda_{n,1} t_n}, \quad \text{sep}(\nu_{(1-c)t_n}^n, \pi^n) \geq 1 - \frac{1}{1 + c^2 \lambda_{n,1} t_n}.$$

The proof is completely analogous to the proof of Theorem 5.1 in [6], and so is omitted.

Example 6.5. Let $0 < 1 = r_1 \leq r_2 \leq r_3 \leq \dots$ be an arbitrary increasing sequence of positive real numbers, and let A_n be the generator of reflecting Brownian motion on $I_n = [0, r_n]$. Then (see [19, Section 6]), we know that

$$\lambda_{n,k} = \frac{j_k^2}{2r_n^2}$$

where $(j_k)_{k=1}^\infty$ are the positive zeros of the usual Bessel function $J_{1/2}$. Note

$$\lambda_{n,1} \mathbb{E} W_n = \sum_{k=1}^\infty \frac{j_1^2}{j_k^2}$$

is constant in n , and therefore there is no separation cut-off.

Example 6.6. Let (η_n) be a sequence of positive real numbers diverging monotonically to infinity. Let A_n be the generator for a $\text{Bes}(2\eta_n+2)$ process on $[0, 1]$ with 1 a reflecting boundary. Again from [19, Section 6], we have that

$$\lambda_{n,k} = \frac{j_{n,k}^2}{2}$$

where $(j_{n,k})_{k=1}^\infty$ are the positive zeros of the Bessel function J_{η_n+1} . Then

$$\lambda_{n,1} \mathbb{E}W_n = \sum_{k=1}^\infty \frac{j_{n,1}^2}{j_{n,k}^2}.$$

It is well known (see for instance [29, equations (1) and (40)]) that

$$\sum_{k=1}^\infty \frac{1}{j_{n,k}^2} = \frac{1}{4(\eta_n + 2)}$$

and (see [1, pg. 371]) that

$$j_{n,1}^2 = \left[\eta_n + 1 + O\left(\eta_n^{-1/3}\right) \right]^2;$$

so there is a separation cut-off for this sequence of diffusions at (t_n) , with $t_n = 2 \sum_{k=1}^\infty j_{n,k}^{-2} = (2\eta_n + 4)^{-1}$.

This is, perhaps, not a surprising result in light of the interpretation of the $\text{Bes}(m)$ process as the radial part of m -dimensional Brownian motion for integer m . As the strong stationary dual of a $\text{Bes}(\alpha)$ process is a $\text{Bes}(\alpha + 2)$ process (recall Example 3.8), for integer sequences $\eta_n = m_n$, a separation cut-off is equivalent to a sharp concentration in the hitting time of 1 of the dual $\text{Bes}(2m_n+4)$ sequence, i.e., a sharp concentration in the hitting time of the unit sphere for $(2m_n+4)$ -dimensional Brownian motion started in $\vec{0}$. For large m_n , at time t the ratio of the square of the radial part of $(2m_n+4)$ -dimensional Brownian motion to t has a distribution which doesn't depend on t and (by the central limit theorem) is approximately normal with mean $2m_n + 4$ and variance $2(2m_n + 4)$. We therefore expect to have a sharp concentration of the hitting time of the unit sphere at $t = (2m_n + 4)^{-1}$, and indeed we found that the cut-off occurs there.

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APPENDIX A. DETAILS OF THEOREM 4.2

In proving Theorem 4.1, we made use of Theorem 4.2 adapted from [8, Theorems 4.8.2 and 1.6.5 and Corollary 4.8.9]]. We restate the theorem here:

Theorem 4.2 *Let A be the generator of a regular diffusion process Y with state space \mathcal{S} . Assume $h_\Delta > 0$ converges to 0 as $\Delta \downarrow 0$. Let $Y_t^\Delta := X_{\lfloor t/h_\Delta \rfloor}^\Delta$ where $X^\Delta \sim (\pi_0^\Delta, P^\Delta)$ is a Markov chain with some metric state space $\mathcal{S}^\Delta \subset \mathcal{S}$, and assume $Y_0^\Delta \Rightarrow Y_0$. Define $T^\Delta : B(\mathcal{S}^\Delta) \rightarrow B(\mathcal{S}^\Delta)$ via*

$$T^\Delta f(x) = \mathbb{E}_x f(X_1^\Delta)$$

for f in the space $B(\mathcal{S}^\Delta)$ of real-valued bounded measurable functions on \mathcal{S}^Δ . Define $A^\Delta := h_\Delta^{-1}(T^\Delta - I)$. Suppose that $C(\mathcal{S})$ is convergence determining and that there is an algebra $B \subset C(\mathcal{S})$ that strongly separates points. Let $\rho_\Delta : C(\mathcal{S}) \rightarrow B(\mathcal{S}^\Delta)$ be defined via $\rho_\Delta f(\cdot) = f|_{\mathcal{S}^\Delta}(\cdot)$. If

$$(B.1) \quad \lim_{\Delta \rightarrow 0} \sup_{y \in \mathcal{S}^\Delta} |(A^\Delta \rho_\Delta f)(y) - (Af)(y)| = 0$$

for all $f \in \mathcal{D}_A$, then $Y^\Delta \Rightarrow Y$.

The purpose of this appendix is to carefully spell out the proof of the above theorem, as the notation in [8] differs considerably from the notation we have adopted. The following chart gives the notational equivalences between the present work and [8]; in connection with $\mu_n(x, \cdot)$, see Corollary 4.8.5 in [8].

Notation in present work:	Notation in [8]:
$\mathcal{S} = [S(0), S(1)]$ with the Euclidean metric	(E, r)
$\mathcal{S}^\Delta, A^\Delta, T^\Delta$	E_n, A_n, T_n
$P^\Delta(x, \cdot)$	$\mu_n(x, \cdot)$
$\{(f, Af) \mid f \in \mathcal{D}_A\}$	A
\mathcal{D}_A	\mathcal{D}_A
$C(\mathcal{S})$	L
$1/h^\Delta$	α_n
id	η_n
ρ_Δ	π_n

Here $\rho_\Delta : C(\mathcal{S}) \rightarrow B(\mathcal{S}^\Delta)$ is defined via $\rho_\Delta f(\cdot) = f|_{\mathcal{S}^\Delta}(\cdot)$, and $\text{id} : \mathcal{S}^\Delta \rightarrow \mathcal{S}$ is the inclusion function embedding \mathcal{S}^Δ into \mathcal{S} .

Proof of Theorem 4.2. As in [8, Section 3.4], define a set $B \subset C(\mathcal{S})$ to be *convergence determining* if

$$\lim_{n \rightarrow \infty} \int f d\mathbb{P}_n = \int f d\mathbb{P} \text{ for all } f \in B$$

implies $\mathbb{P}_n \Rightarrow \mathbb{P}$. We say that $B \subset C(\mathcal{S})$ *strongly separates points* if for every $y \in \mathcal{S}$ and $\varepsilon > 0$ there exists a finite set $\{f_1, \dots, f_k\} \subset B$ such that

$$\inf_{z: |z-y| \geq \varepsilon} \max_{1 \leq i \leq k} |f_i(z) - f_i(y)| > 0.$$

Clearly $C(\mathcal{S})$ is convergence determining, and by considering suitably smooth uniform approximations in \mathcal{D}_A to the indicator function of $\{x\}$ for each $x \in \mathcal{S}$, it follows that $\mathcal{D}_A \subset C(\mathcal{S})$ is an algebra that strongly separates points. In the notation of [8, Corollary 4.8.9], we have $G_n = E_n = \mathcal{S}^\Delta$, and so to prove $Y^\Delta \Rightarrow Y$, it

suffices to prove that for each $T > 0$ and $f \in C(\mathcal{S})$ we have

$$(B.2) \quad \lim_{\Delta \rightarrow 0} \sup_{y \in \mathcal{S}^\Delta} \left| (T^\Delta)^{\lfloor t/h \rfloor} \rho_\Delta f(y) - \rho_\Delta T_t f(y) \right|, \quad 0 \leq t \leq T.$$

From [8, Theorem 1.6.5], to prove (B.2) it suffices to establish that for all $f \in \mathcal{D}_A$ we have that $\rho_\Delta f \in B(\mathcal{S}^\Delta) (= L_n$ in the notation of [8, Theorem 1.6.5]) satisfies

$$(B.3) \quad \lim_{\Delta \rightarrow 0} \sup_{y \in \mathcal{S}^\Delta} |\rho_\Delta f(y) - f(y)| = 0$$

and

$$(B.4) \quad \lim_{\Delta \rightarrow 0} \sup_{y \in \mathcal{S}^\Delta} |(A^\Delta \rho_\Delta f)(y) - (Af)(y)| = 0.$$

But (B.3) is clearly true, and (B.4) is assumed (for all $f \in \mathcal{D}_A$) in the statement of Theorem 4.2. \square

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